AMERICAN

Journal of Mathematics.

SIMON NEWCOMB, Editor.
THOMAS CRAIG, Associate Editor.

PUBLISHED UNDER THE AUSPICES OF THE

JOHNS HOPKINS UNIVERSITY.

Πραγμάτων ἔλεγχος οὐ βλεπομένων.

VOLUME XIII.

BALTIMORE: THE JOHNS HOPKINS PRESS.

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1891.

P14724.

PRESS OF ISAAC FRIEDENWALD CO.
BALTIMORE.

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A. Cayley

Ueber die zu der Curve $\lambda^3\mu + \mu^3\nu + \nu^3\lambda = 0$ *im projectiven Sinne gehörende mehrfache Ueberdeckung der Ebene.*

Von Mellen Woodman Haskell.

Einleitung.

Die geometrische Darstellung von Functionen.

Man gelangt im Allgemeinen auf zwei Weisen zur geometrischen Darstellung des Verlaufs einer algebraischen Function w der Variabelen z, welche durch die Gleichung

$$w = f(z)$$

gegeben wird. Kommt es nur auf reelle Werte an, so repräsentirt man w und z als Coordinaten eines beweglichen Punktes in einer Ebene, und das Aggregat aller solchen Punkte bildet eine algebraische Curve. Diese Darstellungsweise hat den Vorteil, dass die Umkehrbarkeit des Functionenbegriffs sofort daran zu erkennen ist, und dass sie sich ohne Weiteres auch auf implicite Functionen ausdehnen lässt, dass sie also ein Bild der unauflösbaren Gleichung:

$$f(w, z) = 0$$

darbietet.

Will man aber auch die imaginären Werte mit in Betracht ziehen, so bedient man sich der *Riemann'schen Fläche.* Die complexen Grössen $z = x + yi$ werden durch Punkte in einer Ebene dargestellt, deren rechtwinklige Coordinaten x und y sind, und dementsprechend werden die complexen Grössen $w = u + vi$ ebenfalls durch Punkte in einer zweiten Ebene mit den Coordinaten u und v dargestellt. Ist nun w eine n-wertige Function von z, so denkt man sich die z-Ebene als aus n übereinanderliegenden, an bestimmten Stellen zusammenhängenden Blättern bestehend, so dass w auf der auf diese Weise gebildeten Fläche einwertig ist. Hierbei besteht das wichtige Princip der *conformen Abbildung.* Um

die Abbildung eindeutig umzukehren, muss man auch auf der w-Ebene eine Riemann'sche Fläche aufbauen. Die eindeutige Beziehung der beiden Flächen auf einander ist indessen in höheren Fällen nicht mehr übersichtlich, und die Ausdehnung des Verfahrens auf implicite Functionen ist mit einigen Schwierigkeiten verbunden.

Eine Vereinigung dieser beiden Methoden bietet die von Herrn Prof. F. Klein vorgeschlagene *mehrfache Ueberdeckung der Ebene* im projectiven Sinne. Indem ich, was das Nähere betrifft, auf die Arbeiten desselben* verweise, gebe ich hier eine kurze Darlegung des dabei zu Grunde liegenden Princips. Es handelt sich, allgemein zu reden, um eine geometrische Darstellung sämmtlicher Wertsysteme, welche einer Gleichung mit zwei Veränderlichen, bezw. einer homogenen Gleichung mit drei Veränderlichen:

$$f(\lambda, \mu, \nu) = 0$$

genügen.

Im Sinne der gewöhnlichen projectiven Geometrie der Ebene wird jedes Wertsystem $(\lambda:\mu:\nu)$ durch die Coordinaten einer geraden Linie in der Ebene gedeutet. Dieselben werden im Allgemeinen imaginär; es besitzt aber jede imaginäre Gerade einen reellen Punkt, den Schnittpunkt mit der conjugirt imaginären Geraden. Lassen wir jetzt jede imaginäre Gerade durch ihren reellen Punkt vertreten, und beschränken wir uns auf Gerade, deren Coordinaten der Gleichung $f(\lambda, \mu, \nu) = 0$ genügen, so ist jeder Punkt der Ebene im Allgemeinen Vertreter von mehreren imaginären Geraden. Wir denken uns dem zufolge jeden Teil der Ebene mit so vielen Blättern im Riemann'schen Sinne überdeckt, wie die Anzahl der imaginären Geraden beträgt, welche durch die einzelnen Punkte des Ebenenteils vertreten werden. Auf diese Weise bekommen wir eine mehrfache Ueberdeckung der Ebene, bei der jedes Wertsystem $(\lambda:\mu:\nu)$ eindeutig dargestellt wird. Dies Verfahren hat überdies den Vorzug, seinen Charakter bei irgendwelchen reellen Projectivitäten zu behalten.

Beschränken wir uns auf Gleichungen $f = 0$ mit reellen Coefficienten, so vereinfacht sich die Sache. Genügt nämlich der Gleichung ein Wertsystem $(\lambda:\mu:\nu)$, so genügt derselben auch das conjugirt imaginäre Wertsystem $(\lambda':\mu':\nu')$. Alle Vorkommnisse gruppiren sich also paarweise, jeder Punkt der Ebene vertritt eine paare Anzahl gerader Linien, und die Ebene wird in jedem

* F. Klein, "Ueber eine neue Art Riemann'scher Flächen," Mathematische Annalen, Bd. 7 und 10.

Teil von einer geraden Anzahl Blätter überdeckt. Was die reellen Wertsysteme betrifft, so umhüllen die dieselben darstellenden reellen Geraden eine algebraische Curve, (insofern man von reellen isolirten Doppeltangenten absieht, die in dem weiterhin zu betrachtenden Falle nicht vorkommen werden). Jede solche, einem reellen Wertsystem entsprechende reelle Gerade darf als Grenzstelle zweier imaginären Geraden betrachtet werden, deren reeller Schnittpunkt in den Berührungspunkt übergeht (abgesehen von dem Falle einer Wendetangente, die in unserem Falle auch nicht vorkommt). Langs jeden Teils der reellen Curve hängen hiernach zwei Blätter der mehrfachen Ueberdeckung mittelst einer Falte zusammen. Die auf diese Weise gebildete Fläche ist im Allgemeinen ein-eindeutig auf die gewöhnliche Riemann'sche Fläche bezogen. Es fehlt dabei allerdings die Aehnlichkeit in den kleinsten Teilen, wie sie bei der gewöhnlichen Wechselbeziehung zweier Riemann'scher Flächen eintritt. Statt ihrer kommt eine allgemeinere Beziehung, die in einem späteren Kapitel entwickelt werden soll.

Obgleich diese Darstellungsweise für den Verlauf einer algebraischen Function viele Vorteile darbietet, ist sie bis jetzt wenig angewandt worden. Ausser den Arbeiten von Herrn Klein ist nur noch eine Abhandlung von Harnack* zu erwähnen, wo die mehrfache Ueberdeckung für den Fall der Curven dritter Classe studirt worden ist. Es schien daher wünschenswert, die eben erwähnten Principien auf ein etwas complicirteres Beispiel anzuwenden, welches auch unter mannigfachen anderen Gesichtspunkten besonderes Interesse bietet. Dies soll im Folgenden geschehen.

Die vorliegende Untersuchung habe ich auf Veranlassung meines hochverehrten Lehrers Herrn Prof. F. Klein unternommen und unter seiner Leitung ausgeführt. Es handelt sich dabei um die Aufstellung der mehrfachen Ueberdeckung der Ebene, welche zu einer gewissen Normalcurve gehört, die zum Fundamentalpolygon der Modulfunctionen siebenter Stufe in enger Beziehung steht und deren Gleichung und Eigenschaften schon von Herrn Prof. Klein behandelt sind. Von seinen Resultaten ist im Folgenden so viel angegeben, als nötig schien, um das Verständnis dieser Arbeit zu ermöglichen.

Ich möchte es an dieser Stelle nicht unterlassen, Herrn Prof. Klein meinen herzlichsten Dank auszusprechen für die vielfache Anregung und Unterstützung, ohne die es mir nicht möglich gewesen wäre, diese Arbeit zu vollenden.

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*A. Harnack, "Ueber die Verwerthung der elliptischen Functionen für die Geometrie der Curven dritter Classe," Math. Annalen, Bd. 9.

ERSTER ABSCHNITT.

Entstehung der zu untersuchenden Gleichung aus der Theorie der elliptischen Modulfunctionen.

Bei den Untersuchungen über die Transformation siebenter Ordnung der elliptischen Functionen erweisen sich die Modulfunctionen siebenter Stufe als rationale homogene Functionen nullter Dimension zweier Fundamentalfunctionen, die in homogener Bezeichnung $\lambda:\mu:\nu$ heissen sollen, worauf zwischen ihnen die einfache Gleichung besteht:

$$\lambda^3\mu + \mu^3\nu + \nu^3\lambda = 0.$$

Indem ich in diesem Abschnitte den Zusammenhang zwischen der Theorie der elliptischen Modulfunctionen und dieser Gleichung so kurz wie möglich angebe, möchte ich für die im Texte fehlenden Beweise aller Sätze auf die Arbeit von Herrn Klein* über diesen Gegenstand, wovon der folgende Abschnitt hauptsächlich ein blosses Referat bildet, verweisen.

§1.

Das Fundamentalpolygon siebenter Stufe in der τ-Ebene.

In der Theorie der elliptischen Functionen kommt vor allen Dingen in Betracht die rationale absolute Invariante J, welche, als Function des Periodenverhältnisses τ betrachtet, eindeutig ist und bei allen ganzzahligen linearen Substitutionen von τ von der Determinante 1 ungeändert bleibt. Deuten wir diese Beziehung durch eine Riemann'sche Fläche über der J-Ebene, so besteht diese Fläche aus unendlich vielen Blättern, welche bei $J = 0, 1, \infty$ zu je $3, 2, \infty$ zusammenhängen. Nun ist J eine eindeutige Function von τ, die aber nur in der positiven Hälfte der τ-Ebene existirt. (Die Axe des Reellen in der τ-Ebene bildet nämlich eine natürliche Grenze für die Function $J(\tau)$). Es ist daher möglich, den Bereich, innerhalb dessen J existirt, d. h., die positive Halbebene, in unendlich viele nebeneinanderliegende Gebiete einzuteilen, deren jedes auf ein Blatt der Riemann'schen Fläche abgebildet ist. Diese Gebietseinteilung ist sehr viel übersichtlicher als die vorgenannte Riemann'sche Fläche

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*F. Klein, "Ueber die Transformation siebenter Ordnung der elliptischen Functionen," Math. Annalen, Bd. 14.

über der J-Ebene und wird also im Folgenden statt ihrer durchweg zu Grunde gelegt. Sie geschieht durch unendliche Vervielfältigung der Figur 1 vermittelst Combination und Wiederholung der Fundamentalsubstitutionen,

$$S:\ \tau' = \tau + 1\,,$$
$$T:\ \tau' = -\frac{1}{\tau}\,.$$

Es stellt die Substitution S eine Parallelverschiebung, die Substitution T eine Transformation durch reciproke Radien gegen den Einheitskreis verbunden mit einer Spiegelung gegen die Axe des Imaginären dar.

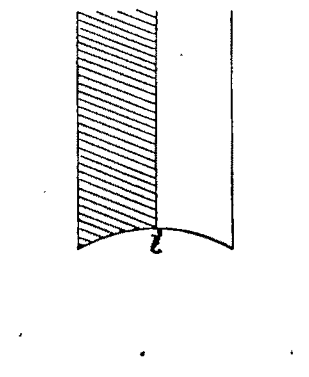

Fig. 1.

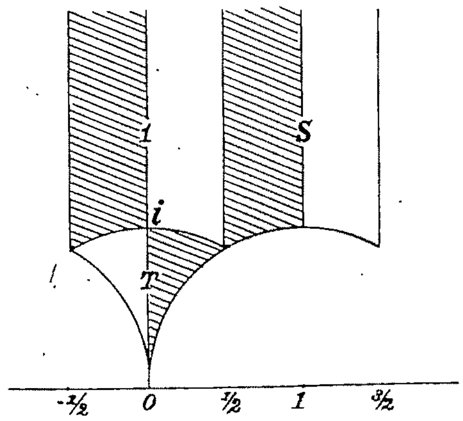

Fig. 2.

Die geometrische Bedeutung der beiden Substitutionen wird in Fig. 2 geschildert, indem die Gebiete, welche aus dem fundamentalen Gebiet (1) hervorgehen, einfach mit den Buchstaben S, resp. T bezeichnet sind. Dabei entspricht jedes schraffirte Kreisbogendreieck einem positiven Halbblatt der J-Ebene, jedes nicht schraffirte Dreieck einem negativen Halbblatt, und die drei Dreiecksseiten entsprechen je drei verschiedenen Stücken der reellen Axe der J-Ebene.

So viel über die Beziehung zwischen J und τ! Bei der Transformation siebenter Ordnung werden nun auch noch solche eindeutige Functionen η von τ in Betracht gezogen, welche bei denjenigen τ-Substitutionen, die modulo 7 zur Identität congruent sind, und nur bei solchen, ungeändert bleiben,—die Modulfunctionen siebenter Stufe der allgemeinsten Art. Nun verteilen sich die τ-Sub-

stitutionen modulo 7 auf 168 verschiedene Classen, und η nimmt also bei der Gesammtheit derselben im Ganzen 168 Werte an. Es ist also η eine 168-wertige Function von J. Man kann die Verzweigung der Function η in Bezug auf J durch eine Riemann'sche Fläche über der J-Ebene darstellen, die aus 168 Blättern besteht, welche bei $J = 0, 1, \infty$ zu je 3, 2, 7 zusammenhängen. Diesen dreierlei Verzweigungspunkten werden, ihrer Wichtigkeit halber, besondere Namen zuerteilt, und es wird von jetzt an immer die Rede sein von den 24 Punkten a, wo die Blätter zu je 7 zusammenhängen, sodann von den 56 Punkten b und von den 84 Punkten c, wo die Blätter zu je 3, resp. zu je 2 zusammenhängen. Das Geschlecht der Fläche ist also:

$$p = \frac{1}{2}(2 - 2.168 + 56.2 + 84.1 + 24.6) = 3.$$

Mit dieser vielblättrigen Fläche ist aber gar nichts anzufangen, da sie durchaus unübersichtlich ist, und man greift also zu dem Hilfsmittel hin, welches die Gebietseinteilung der τ-Ebene bietet. *Es ist nämlich möglich, die Riemann'sche Fläche auf ein begrenztes Stück der τ-Ebene abzubilden, dessen Ränder in bestimmter Weise zusammengehören.* Um diese Abbildung kurz zu bezeichnen, sollen zunächst solche Gebiete in der τ-Ebene, welche bei den modulo 7 zur Identität congruenten Substitutionen aus einander hervorgehen, als congruent, und dementsprechend solche Gebiete, die nicht bei solchen Substitutionen aus einander hervorgehen, als incongruent bezeichnet werden. Nun ist jedes Doppeldreieck der τ-Ebene eine conforme Abbildung der J-Ebene; man kann also ein Aggregat von 168 incongruenten Doppeldreiecken als ein vollständiges Bild der 168-blättrigen Riemann'schen Fläche betrachten, wenn noch angegeben wird, wie die Kanten der 168 Doppeldreiecke zusammengehören. Diese Zusammengehörigkeit muss sich durch Substitutionen ausdrücken, die modulo 7 zur Identität congruent sind. Eine solche Figur, die aus einem einzigen zusammenhängenden Stück der τ-Ebene besteht, entsteht zum Beispiel wenn 14 abwechselnd symmetrische, der Figur 3 ähnliche Figuren neben einander aufgestellt werden.

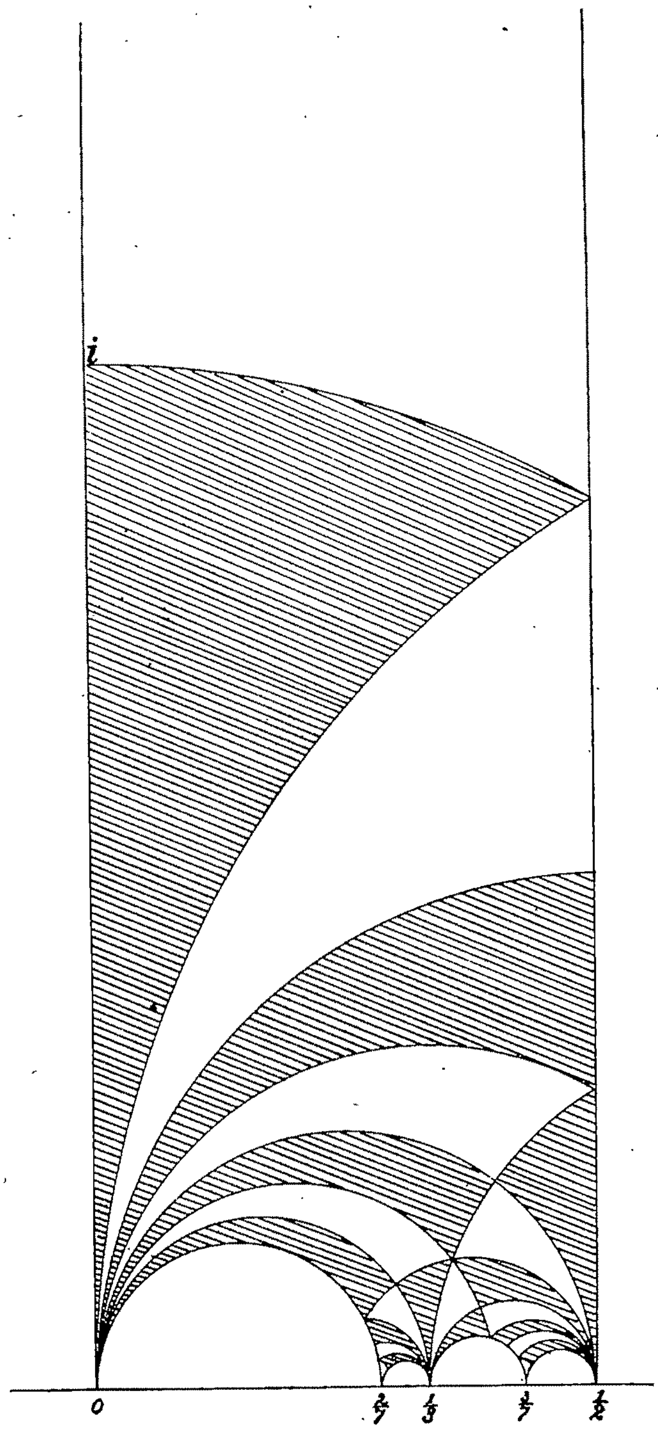

FIG. 3.

Die Zusammengehörigkeit der Kanten wird durch "congruente" Substitutionen vermittelt. Es folgt, dass die beiden Geraden, welche die Figur rechts und links abgrenzen, zusammengehören, denn sie gehen bei der Substitution: $\tau' = \tau + 7$ in einander über. Ferner gehören jedesmal zusammen diejenigen beiden begrenzenden Halbkreise, die in den Punkten $\tau = 0, \frac{1}{2}, 1$, etc., zusammenstossen, wobei die genannten Punkte immer sich selbst entsprechen. Um die Zusammengehörigkeit irgend zweier Kanten auszudrücken, genügt es, von zwei Punkten der einen Kante anzugeben, in welche zwei Punkte der anderen Kante sie übergehen. Die Zusammengehörigkeit der übrigen begrenzenden Halbkreise der Figur wird also in folgender Tabelle angegeben, wo jedesmal die betreffenden Halbkreise durch ihre beiden Endpunkte angedeutet werden. Hierbei gehören immer zusammen die in der Tabelle gerade übereinanderliegenden Endpunkte, beispielsweise $\frac{2}{7}$ und $\frac{19}{7}$, $\frac{1}{3}$ und $\frac{8}{3}$.

$\frac{2}{7}, \frac{1}{3}$	$\frac{1}{3}, \frac{8}{7}$	$\frac{9}{7}, \frac{4}{3}$	$\frac{4}{3}, \frac{10}{7}$	$\frac{16}{7}, \frac{7}{3}$	$\frac{7}{3}, \frac{17}{7}$	$\frac{23}{7}, \frac{10}{3}$
$\frac{19}{7}, \frac{8}{3}$	$\frac{8}{3}, \frac{18}{7}$	$-\frac{23}{7}, -\frac{10}{3}$	$-\frac{10}{3}, -\frac{24}{7}$	$-\frac{16}{7}, -\frac{7}{3}$	$-\frac{7}{3}, -\frac{17}{7}$	$-\frac{9}{7}, -\frac{4}{3}$
$\frac{10}{3}, \frac{24}{7}$	$-\frac{19}{7}, -\frac{8}{3}$	$-\frac{8}{3}, -\frac{18}{7}$	$-\frac{12}{7}, -\frac{5}{3}$	$-\frac{5}{3}, -\frac{11}{7}$	$-\frac{5}{7}, -\frac{2}{3}$	$-\frac{2}{3}, -\frac{4}{7}$
$-\frac{4}{3}, -\frac{10}{7}$	$-\frac{2}{7}, -\frac{1}{3}$	$-\frac{1}{3}, -\frac{3}{7}$	$\frac{5}{7}, \frac{2}{3}$	$\frac{2}{3}, \frac{4}{7}$	$\frac{12}{7}, \frac{5}{3}$	$\frac{5}{3}, \frac{11}{7}$

Die hiermit beschriebene Figur der τ-Ebene soll fortan das Fundamentalpolygon in der τ-Ebene genannt werden. Denken wir uns die Ränder desselben in angegebener Weise verbunden, so haben wir damit ein eindeutiges Bild der Riemann'schen Fläche für $\eta(J)$. Den Verzweigungspunkten der letzteren entsprechend, stossen die Dreiecke des Fundamentalpolygons zu je 2.7, 2.3, 2.2 in 24 Punkten a, bezw. in 56 b und in 84 c zusammen. Wendet man nun irgend eine der τ-Substitutionen auf das Fundamentalpolygon an, und denkt man sich von den resultirenden Dreiecken alle, die ausserhalb fallen, durch die ihnen äquivalenten ersetzt, so geht dabei das Fundamentalpolygon in sich über. In diesem Sinne lässt sich das Fundamentalpolygon auf 168 Weisen in sich transformiren, wobei die Transformationen, welche den Substitutionen S und T entsprechen, und die man also als erzeugende Operationen der 168 betrachten darf, die Perioden 7, resp. 2 besitzen.

Bei genauerem Studium der zu dieser Riemann'schen Fläche, resp. zu diesem Fundamentalpolygon gehörenden algebraischen Functionen ergeben sich nun sämmtliche solche Functionen als rationale homogene Functionen nullter Dimen-

sion von zwei Normalfunctionen $\lambda:\mu:\nu$, welche durch die Gleichung vierter Ordnung:

$$\lambda^3\mu + \mu^3\nu + \nu^3\lambda = 0$$

verbunden sind. Diese Gleichung und ihre Beziehung zu τ und zu J geometrisch anschaulich zu untersuchen, ist die Aufgabe der vorliegenden Arbeit.

§2.

Uebertragung des Fundamentalpolygons auf die s-Ebene.

Der grösseren Uebersichtlichkeit wegen führen wir zunächst noch ein neues Hilfsmittel ein. Wir können die Dreiecke des Fundamentalpolygons auf Kreisbogendreiecke mit den Winkeln $\frac{\pi}{7}, \frac{\pi}{2}, \frac{\pi}{3}$ conform abbilden.* Lassen wir dieselben nach dem Princip der Symmetrie aufeinanderfolgen, so bekommen wir bei geeigneter Wahl des Ausgangsdreiecks das Fundamentalpolygon auf eine solche Figur, wie sie in Tafel I angegeben ist, abgebildet, wobei die Zusammengehörigkeit der noch freien Kanten in der danebenstehenden Tabelle angegeben ist. Die einzelnen Doppeldreiecke gehen bekanntlich wieder durch lineare Transformation aus einander hervor. Diese Figur darf *das Fundamentalpolygon in der s-Ebene* genannt werden. Sie ist, was Uebersichtlichkeit betrifft, dem τ-Polygon überlegen, weil sie einen höheren Grad von Symmetrie besitzt, weil sie ganz im Endlichen gelegen ist, und endlich, wie sofort gezeigt werden soll, weil die geometrische Bedeutung der vorgenannten 168 Transformationen der Riemann'schen Fläche in sich selbst im höchsten Grade zur Anschauung gebracht wird. Lassen wir etwa den Mittelpunkt A der Figur dem unendlich fernen Punkt des τ-Polygons entsprechen, so ist die Bedeutung der Substitution von der Periode 7, die wir S genannt haben, sofort klar. Sie stellt eine Drehung um A durch den Winkel $\frac{2\pi}{7}$ dar. Entspricht nun B dem Punkt $\tau = i$, so wirkt die Substitution T in der s-Ebene gerade so wie in der τ-Ebene. Sie stellt nämlich eine Combination der Kreisverwandtschaft in Bezug auf den Kreis durch B und der Spiegelung gegen die Mittellinie der Figur dar. Diese Operation

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* Eine solche Abbildung wird durch die Schwarz'sche s-Function: $s\left(\frac{1}{3}, \frac{1}{2}, \frac{1}{7}, J\right)$ vermittelt. Cf. Schwarz, "Ueber diejenigen Fälle, in welchen die Gauss'sche hypergeometrische Reihe, etc." Crelle's Journal, Bd. 75.

können wir kurzweg als eine *Drehung im projectiven Sinne* um den Punkt B bezeichnen; sie reducirt sich auf eine gewöhnliche Drehung, wenn wir die Ebene einer solchen Transformation unterwerfen, bei welcher der Kreis durch B in eine Gerade verwandelt wird. Um der Substitution T eine Bedeutung für das Fundamentalpolygon in der s-Ebene zu geben, müssen wir, gerade wie bei dem τ-Polygon, jedes Dreieck, welches nach Anwendung der Substitution T ausserhalb des Fundamentalpolygons fällt, durch das innerhalb desselben liegende äquivalente congruente Dreieck ersetzt denken. In diesem Sinne können wir wieder von Transformationen des Fundamentalpolygons in sich selbst reden, bezw. von Drehungen in projectiven Sinne, welche das Fundamentalpolygon mit sich selbst zur Deckung bringen.

Es ist nicht schwer, die Bedeutung der übrigen Substitutionen aus der Figur abzulesen, indem man aus einem Doppeldreieck durch Combination und Wiederholung der den Substitutionen S und T entsprechenden Operationen die übrigen Doppeldreiecke allmählich entstehen lässt. Ich habe diese Substitutionen, da ich sie später mehrfach gebrauchen werde, in die Figur des Fundamentalpolygons hineingeführt, indem ich in jedes Dreieck diejenige Operation hineingeschrieben habe, durch welche dasselbe aus dem mit (1) bezeichneten Dreieck entsteht. Um die Bedeutung einer solchen Substitution recht klar zu haben, wolle man noch beachten, dass dabei die Punkte a immer in einander übergehen, desgleichen die Punkte b, resp. c, und anliegende Dreiecke in anliegende Dreiecke.

Man denke sich nun statt des Fundamentalpolygons der s-Ebene am Besten gleich die geschlossene Fläche im Raume, welche entsteht, indem man sich die Ränder des Polygons in der angegebenen Weise zusammengeheftet denkt. Sie hat dann 168 eindeutige Transformationen in sich, die man sich leicht als projective Drehungen vorstellen kann.

Wir können nun aus der Figur eine wichtige Folge für die Gleichung:

$$\lambda^3\mu + \mu^3\nu + \nu^3\lambda = 0 \tag{1}$$

ziehen. Bei den 168 Substitutionen nehmen die einzelnen Wertsysteme $\lambda:\mu:\nu$ im Allgemeinen 168 verschiedene Werte an. Die Gleichung (1) bleibt aber immer bestehen. Sie ist also eine Gleichung mit 168 eindeutigen Transformationen in sich. Den Punkten a, b, c entsprechend, muss es auch drei Gruppen von 28, resp. 56 und 84 Wertsystemen $\lambda:\mu:\nu$ geben, welche bei den 168 Transformationen der Gleichung in sich in einander übergehen. Wir führen kurz an, dass

diese 168 Transformationen *lineare* Substitutionen der $\lambda:\mu:\nu$ sind. Dies ist für das Folgende besonders wichtig.

Ehe wir nun zur genaueren Betrachtung der Gleichung (1) übergehen, wollen wir erst im Fundamentalpolygon gewisse Linien definiren, die wir später gebrauchen,—*die* 28 *Symmetrielinien.*

1. *Eine Symmetrielinie wird aus solchen Dreiecksseiten gebildet, die auf dem geschlossenen Polygon ohne Knickung auf einander folgen.* Dabei ist zu verstehen, auch an den Stellen, wo die Linie über den Rand des Polygons hinübersetzt, dass eine Linie ohne Knickung verläuft, wenn die Summe der Dreieckswinkel auf der einen Seite der Linie gleich der Summe der Dreieckswinkel auf der anderen Seite ist.

2. Eine solche Symmetrielinie bildet zum Beispiel die Mittellinie der Figur, zusammengenommen mit ihrer Verlängerung an den Kanten 5, resp. 10. Dass diese Curven wirklich eine geschlossene Curve auf dem geschlossenen Polygon bilden, findet seine Bestätigung in den kleinen Nebenfiguren, welche den Zusammenhang des Polygons in den Ecken der einen, resp. der anderen Art schildern.

3. In Bezug auf diese Curve ist die Figur offenbar symmetrisch im gewöhnlichen Sinne, denn sie liegt an sich in der s-Ebene Punkt für Punkt symmetrisch, und übrigens, wie aus der bezüglichen Tabelle hervorgeht, ist auch die Verbindungsweise der Kanten in Bezug auf diese Linie durchaus symmetrisch.

4. Aus dem Bildungsgesetz der Symmetrielinien ist sofort klar, dass bei Anwendung irgend einer der 168 Substitutionen aus dieser ersten Symmetrielinie immer eine zweite hervorgeht, die im Allgemeinen eine kreisförmige Gestalt haben wird, bezw. aus kreisförmigen Stücken bestehen wird. Im besonderen Falle kann das einzelne neue Stück geradlinig werden. Da bei einer solchen Substitution das Fundamentalpolygon in sich selbst transformirt wird, so ist die Figur auch in Bezug auf die zweite Symmetrielinie symmetrisch, d. h. symmetrisch im projectiven Sinne.

5. In der Gruppe von 168 Substitutionen sind diese symmetrischen Umformungen des Fundamentalpolygons nicht einbegriffen, wohl aber die Combination derselben zu je 2. Es gibt aber einen Gesichtspunkt, unter welchem sich die zweierlei Operationen zusammenfügen. Nehmen wir nämlich auch solche Umformungen in Betracht, die schraffirte Dreiecke in nicht schraffirte Dreiecke überführen, so bekommen wir eine *erweiterte Gruppe* von 2.168 Substitutionen, innerhalb deren die G_{168} als ausgezeichnete Untergruppe enthalten ist. Unter

diesen Substitutionen finden sich unsere symmetrischen Umformungen mit vor. Dieselben sind solche Substitutionen unter den 2.168, bei denen jedesmal eine Symmetrielinie Punkt für Punkt in sich übergeht.

6. Aus diesen Betrachtungen sehen wir, dass die Symmetrielinien gegenüber der G_{168} *gleichberechtigt* sind, so dass wir immer nur eine zu betrachten haben, um Sätze über alle zu erhalten. Als die einfachste Symmetrielinie wählen wir die oben beschriebene, die wir als *Grundlinie* bezeichnen wollen.

7. Die Haupteigenschaft der Symmetrielinien ist nun die folgende, die sich sofort an der Grundlinie constatiren lässt. Die auf einer Symmetrielinie liegenden Punkte a, b, c haben eine ganz bestimmte Aufeinanderfolge. *Es sind von jeder Sorte jedesmal sechs, deren Reihenfolge durch die dreimalige Wiederholung des folgenden Schemas angegeben wird:*

$$(a\,b\,c\;b\,a\,c).$$

8. Hieraus ergibt sich sofort die Anzahl der Symmetrielinien. Es liegen nämlich auf jeder Linie 6 Punkte jeder Art; durch jeden Punkt a laufen 7 Linien, durch jeden b 3, durch jeden c 2. Die Anzahl der Symmetrielinien ist also

$$n = \frac{24.7}{6} = \frac{56.3}{6} = \frac{84.2}{6} = 28.$$

Es wird wohl auch von Nutzen sein, noch den Verlauf einer zweiten, in der Figur etwas complicirteren Symmetrielinie genau anzugeben. Wählen wir also beispielsweise die Linie, wovon der eine Teil die Punkte F und G auf den Kanten 12, resp. 3 verbindet. Dieselbe läuft von F nach G, dann, da die Kanten 3 und 8 zu verbinden sind und G und G' daher zusammenfallen, von G' nach H, dann, da 5 und 10 zu verbinden sind, von H' nach F', und F' fällt mit F zusammen. In ähnlicher Weise sind alle anderen Symmetrielinien aus den verschiedenen Kreisbogenstücken des Fundamentalpolygons in der s-Ebene zusammenzusetzen.

§3.

Aufzählung der Symmetrielinien unter Zugrundelegung von einer derselben.

Da es 28 Symmetrielinien gibt, so bleibt jede derselben bei 6 Operationen der G_{168} ungeändert. Wir wollen dieselben bei der Grundlinie aufzählen, womit im Princip die Aufzählung für alle anderen mit gemacht ist; denn wegen der Gleichberechtigung der Symmetrielinien entsteht immer eine solche Untergruppe

von 6 Operationen durch Transformation einer ersten solchen vermittelst irgend einer Substitution der G_{168}.

Durch einfache Betrachtung der Figur ergibt sich sofort das Resultat: die Untergruppe von 6 Operationen, bei der die Grundlinie in sich übergeht, entsteht durch Combination und Wiederholung der beiden erzeugenden Substitutionen:

$$T\text{(von der Periode 2)},$$
$$U\text{(von der Periode 3)},$$

wo ich der Kürze halber die Substitution $TS^3\,TS^4\,TS^2$ mit U bezeichnet habe.

Die 6 Operationen der betreffenden Untergruppe heissen dann

$$1,\ U,\ U^2,\ T,\ TU,\ TU^2,$$

von denen die drei letzten die Periode 2 haben. Bei den Substitutionen der Periode 2 bleiben jedesmal zwei Punkte c der Grundlinie fest, während sich die Grundlinie, so zu sagen, um dieselben dreht, so dass sich die beiden Teile, in welche sie durch die Punkte c zerlegt wird, gegen einander vertauschen. Bei den Substitutionen von der Periode 3 wird die Grundlinie um ein Drittel ihrer Länge verschoben (projectivisch zu verstehen).

Es ist nun nützlich, die übrigen 27 Symmetrielinien gegenüber dieser Grundlinie zu gruppiren, wobei die genannte G_6 zur Geltung kommt. Hierbei erhalten wir folgende Aufzählung:

I. *Die drei dünn ausgezogenen Linien.* Die übrigen Symmetrielinien liegen im Allgemeinen in Bezug auf die Grundlinie paarweise symmetrisch. Drei aber, die dünn ausgezogenen Linien der Figur, sind mit sich selbst symmetrisch. Sie schneiden die Grundlinie je in zwei solchen Punkten c, die bei einer Substitution der Periode 2 fest bleiben, und alle drei schneiden einander in zwei symmetrisch liegenden Punkten b. Diese beiden Punkte b bleiben bei der genannten Substitution U von der Periode 3 fest, während die drei Symmetrielinien die wir betrachten sich dabei cyclisch permutiren. In der That dürfte das Verhalten der ganzen Figur bei dieser Substitution als eine Drehung (im projectiven Sinne) von der Periode 3 um diese beiden Punkte b herum betrachtet werden. Den Verlauf der einen Linie dieser Art haben wir schon studirt. Der Verlauf der anderen ergibt sich leicht unter Benutzung der Tabelle für die Zusammengehörigkeit der Kanten des Polygons.

Von den verschiedenen Arten von Symmetrielinien greifen wir jedesmal eine als Beispiel heraus, denn das Verhalten der anderen ist an der einen leicht

zu ersehen, und nennen einige der Substitutionen, bei welchen dieselbe aus der Grundlinie hervorgeht. In diesem Sinne wird beispielsweise diejenige, deren Verlauf wir schon studirt haben, bei den Substitutionen S^4TS und S^3TS^6 aus der Grundlinie hervorgehen.

II. Die zweite Art besteht aus drei Paaren beziehungsweise symmetrisch liegender Linien, von denen jedes Paar durch zwei Punkte b der Grundlinie hindurchlaufen. Dieselben sind in der Figur punktirt. Das eine Paar geht aus der Grundlinie beispielsweise bei den Substitutionen S^3TS^3, resp. S^4TS^4 hervor. Bei den Transformationen der G_6 gehen diese Curven paarweise in einander über.

III. Die Linien der dritten Art (in der Figur strichpunktirt) schneiden die Grundlinie gar nicht. Sie verteilen sich aber auf drei Paare symmetrisch liegender Curven, welche bei der G_6 in einander übergehen. Von diesen Linien schneidet sich jedes Paar zusammengehöriger in zwei zur Grundlinie symmetrischen Punkten c. Dieser Umstand wird später von Interesse sein. Das eine Paar geht aus der Grundlinie beispielsweise bei den Substitutionen S^3TS, resp. S^4TS^6 hervor.

IV. Die Linien der vierten Art (in der Figur gestrichelt) verteilen sich auf zwei Gruppen von je sechs, die aber keinen wesentlichen Unterschied von einander erweisen. Die erste dieser Gruppen besteht aus den sechs Geraden, welche ausser der Grundlinie durch den centralen Punkt A der Figur hindurchlaufen, nebst den zugehörigen Verlängerungen an den Kanten der Figur. Jede dieser Linien ergibt sich aus einfacher Drehung der ganzen Grundlinie um den Punkt A, also durch Anwendung der Substitutionen $S, S^2, \ldots . S^6$. Sie schneiden sich auf der Grundlinie in zwei weiteren Punkten a, und zwar in denjenigen, die bei den in der G_6 enthaltenen Substitutionen von der Periode 3 aus A hervorgehen. Sonst schneiden sie weder die Grundlinie noch einander.

Die zweite Gruppe besteht aus den sechs Linien, welche sämmtlich durch die übrigen 3 Punkte a der Grundlinie hindurchlaufen, und aus der Grundlinie bei den Substitutionen $ST, S^2T, \ldots . S^6T$ hervorgehen.

Gegenüber der G_6 spalten sich diese 12 Linien auf etwas andere Weise; doch ist die Spaltung wieder in zwei Gruppen von je sechs, und zwar enthält die erste Sechs drei Curven, welche durch die ersten drei Punkte a gehen, sowie drei Curven durch die zweiten drei Punkte a, während die zweite Sechs diejenigen sechs Curven enthält, die zu denen der ersten Sechs bezüglich der Grundlinie symmetrisch sind.

§4.

Festlegung der zum Fundamentalpolygon gehörigen Normalgleichung.

Wie oben schon erwähnt wurde, reducirt sich die Untersuchung aller zum Fundamentalpolygon gehörigen Functionen auf die Betrachtung zweier Fundamentalfunctionen $\lambda:\mu:\nu$, zwischen denen die algebraische Relation besteht:

$$\lambda^3\mu + \mu^3\nu + \nu^3\lambda = 0.$$

Alle im Fundamentalpolygon eindeutigen Functionen ohne wesentlich singuläre Punkte sind nämlich rationale Functionen der Verhältnisse $\lambda:\mu:\nu$. In diesem Sinne bilden $\lambda:\mu:\nu$ ein *volles System* der zugehörigen Functionen.

Die angegebene Gleichung wird wegen ihrer Einfachheit die *Normalgleichung* genannt. Dieselbe ist, wie in §2 gesagt wurde, eine Gleichung mit 168 linearen Substitutionen in sich. Diese Gruppe von 168 Substitutionen entsteht durch Wiederholung und Combination der beiden folgenden erzeugenden Substitutionen, welche den τ-Substitutionen entsprechend auch mit S und T bezeichnet werden mögen:

S (von der Periode 7):

$$\lambda' = \gamma\lambda,\ \mu' = \gamma^4\mu,\ \nu' = \gamma^2\nu,$$

T (von der Periode 2):

$$\begin{aligned} \lambda' &= A\lambda + B\mu + C\nu, \\ \mu' &= B\lambda + C\mu + A\nu, \\ \nu' &= C\lambda + A\mu + B\nu, \end{aligned}$$

wobei gesetzt wird:

$$\gamma = e^{\frac{2i\pi}{7}},\ A = \frac{\gamma^5 - \gamma^2}{\sqrt{-7}},\ B = \frac{\gamma^3 - \gamma^4}{\sqrt{-7}},\ C = \frac{\gamma^6 - \gamma}{\sqrt{-7}}.^*$$

Die Aufgabe der Untersuchung der Normalgleichung ist zunächst eine rein algebraische. Es wird aber kürzer und bequemer sein, wenn wir die dabei auftretenden Sätze gleich in geometrischer Sprechweise formuliren. Deuten wir also $\lambda:\mu:\nu$ als Punktcoordinaten in der Ebene, so stellt die Gleichung *eine Curve vierter Ordnung ohne Doppelpunkt und ohne Rückkehrpunkt*, also eine Curve mit 24 Wendepunkten und 28 Doppeltangenten dar. Wir berichten nun weiter, dass die Punkte a, b, c des Fundamentalpolygons eine sehr einfache Bedeutung haben. Den 24 Punkten a entsprechen nämlich die 24 Wendepunkte, den 56

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* In den späteren Berechnungen werden folgende identische Relationen zwischen den Grössen A, B, C nützlich sein:

$$A+B+C+1=0,\qquad AB+BC+CA=0,$$
$$A^3B+B^3C+C^3A=0,\qquad A^4B+B^4C+C^4A=0,$$
$$7ABC=1.$$

Punkten b entsprechen die 56 Berührungspunkte der 28 Doppeltangenten. Endlich besitzt die Curve 84 sogenannte sextaktische Punkte, und dieselben entsprechen den 84 Punkten c. Es werden in der Folge der Bequemlichkeit halber diese merkwürdigen Punkte auf der Curve ebenfalls mit den Namen a, b, c bezeichnet.

Was die Configuration der merkwürdigen Punkte angeht, so sind folgende Sätze von besonderer Wichtigkeit:

1. Die 24 Punkte a (die Wendepunkte) verteilen sich auf 8 Tripel, indem jedesmal die 3 Punkte eines Tripels die Ecken eines Dreiecks sind, dessen Seiten die zugehörigen Wendetangenten bilden.

2. Unter den Collineationen der Curve in sich gibt es 21 von der Periode 2, welche also Perspectivitäten sind. Jede Perspectivitätsaxe schneidet die Curve in 4 Punkten, welche bei der betreffenden Collineation festbleiben, und diese $21 \cdot 4 = 84$ Punkte sind eben die 84 sextaktischen Punkte c.

Wir wollen endlich noch in diesem Kapitel die Formeln angeben, welche die Abhängigkeit der $\lambda:\mu:\nu$ von τ ausdrücken. Dieselbe drückt sich durch ϑ-Reihen aus. Um nicht zu viele neue Bezeichnungen einzuführen, teile ich dieselben in folgender einfachster Form mit:

$$\rho\lambda = -q^{\frac{1}{7}}\sum_{h=0}^{\infty}(-1)^h q^{7h(h+1)}\left[q^{(2h+1)} - q^{-(2h+1)}\right],$$

$$\rho\mu = q^{\frac{4}{7}}\sum_{h=0}^{\infty}(-1)^h q^{7h(h+1)}\left[q^{2(2h+1)} - q^{-2(2h+1)}\right],$$

$$\rho\nu = q^{\frac{16}{7}}\sum_{h=0}^{\infty}(-1)^h q^{7h(h+1)}\left[q^{4(2h+1)} - q^{-4(2h+1)}\right]$$

wobei $q = e^{i\pi\tau}$ und ρ ein Proportionalitätsfactor ist.

Zweiter Abschnitt.

Deutung der reellen Elemente der Gleichung durch eine Curve in der Ebene.

Als Vorbereitung zu der späteren Entwickelung der in der Einleitung beschriebenen mehrfachen Ueberdeckung der Ebene sollen jetzt die reellen Elemente der Gleichung:

$$\lambda^3\mu + \mu^3\nu + \nu^3\lambda = 0 \tag{1}$$

festgestellt werden und die geometrische Deutung derselben, zunächst im Sinne der Punktgeometrie, dann aber in engerer Beziehung zu dem Folgenden im Sinne der Liniengeometrie gegeben werden.

§5.

Der reelle Curvenzug.

Bei der folgenden Untersuchung können wir einerseits von der Gleichung (1), andererseits von der Darstellung von $\lambda:\mu:\nu$ durch Potenzreihen, die nach $q = e^{i\pi\tau}$ fortschreiten, ausgehen. Schreiben wir dieselben in der zu der Berechnung zweckmässigen Form:

$$\begin{aligned}
\rho\lambda &= -\,q^{\frac{8}{7}}(-q^{-2} + 1 + q^{10} - \ldots.),\\
\rho\mu &= \quad q^{\frac{4}{7}}(-q^{-2} + q^{2} + q^{8} - \ldots.);\\
\rho\nu &= \quad q^{\frac{16}{7}}(-q^{-4} + q^{2} + q^{4} - \ldots.),
\end{aligned}$$

so ist zunächst klar, dass die Werte $\tau = a + bi$ und $\tau = -a + bi$ conjugirt imaginäre Werte von q, also auch conjugirt imaginäre Werte von $\lambda:\mu:\nu$ ergeben. Es werden also nur solche Werte $\lambda:\mu:\nu$ reelle sein, welche bei Spiegelung gegen die Axe des Imaginären in der τ-Ebene ungeändert bleiben. Nun betrachte man das in der τ-Ebene gelegene Fundamentalpolygon, welches die Gesammtheit der Wertsysteme $\lambda:\mu:\nu$ wiedergibt. Offenbar müssen jedenfalls die Punkte der Mittellinie (Axe des Imaginären) reelle $\lambda:\mu:\nu$ ergeben.

Ausserdem ergeben reelle $\lambda:\mu:\nu$ solche Randstücke des Polygons, welche mit denjenigen Randstücken, die in Bezug auf die Mittellinie ihre Spiegelbilder sind, gerade zusammengeordnet sind. Es sind dies, wie aus Figur 3, resp. aus der Tabelle auf Seite 13 zu ersehen, nur bei folgenden Stücken der Fall:

1. Die beiden begrenzenden Halbkreise, welche im Punkte $\tau = 0$ mit der Mittellinie zusammenstossen;

2. Die beiden begrenzenden verticalen Geraden durch die Punkte $\tau = \frac{7}{2}$, resp. $\tau = -\frac{7}{2}$;

3. Die begrenzenden Halbkreise durch dieselben Punkte;

4. Die zwei Paare begrenzender Halbkreise, deren Diameter die Strecken sind: $\pm\frac{16}{7}$ bis $\pm\frac{7}{3}$, und $\pm\frac{7}{3}$ bis $\pm\frac{17}{7}$.

Das sind nun gerade diejenigen Stücke, die mit der Mittellinie zusammen nach unseren früheren Angaben, wie der Vergleich der Figuren der τ-Ebene und der s-Ebene zeigt, die eine Symmetrielinie, die wir als Grundlinie genommen haben, ausmachen. Wir haben daher folgenden

SATZ: *Die Punkte der von uns bereits ausgezeichneten Symmetrielinie ergeben sämmtlich reelle Werte von $\lambda:\mu:\nu$, und sie sind die einzigen Punkte, die diese Eigenschaft besitzen.*

In dieser Thatsache liegt auch der Grund dafür, dass wir diese Symmetrielinie als Grundlinie angenommen haben. Unsere Curve hat also einen und nur einen reellen Curvenzug. Derselbe entspricht genau der Symmetrielinie, und muss also eine geschlossene Curve bilden; ferner müssen merkwürdige Punkte a, b, c in der bekannten Anzahl und Aufeinanderfolge, welche durch die dreimalige Wiederholung des Schemas:

$$(a\,b\,c\;\;b\,a\,c)$$

angegeben wird, darauf liegen.

Um jetzt diesen einen reellen Curvenzug zu zeichnen, folgen wir dem Princip. Gleichberechtigtes auch für das Auge möglichst gleichartig hervortreten zu lassen. Wir wählen also das Coordinatendreieck (λ, μ, ν) als gleichseitiges Dreieck und die $\lambda:\mu:\nu$ mit den Abständen von den Dreiecksseiten selbst proportional, wobei $\lambda + \mu + \nu = 0$ die Gleichung der unendlich fernen Geraden ist. Dann erhält man für den reellen Curvenzug die einfache Gestalt der Figur 4.

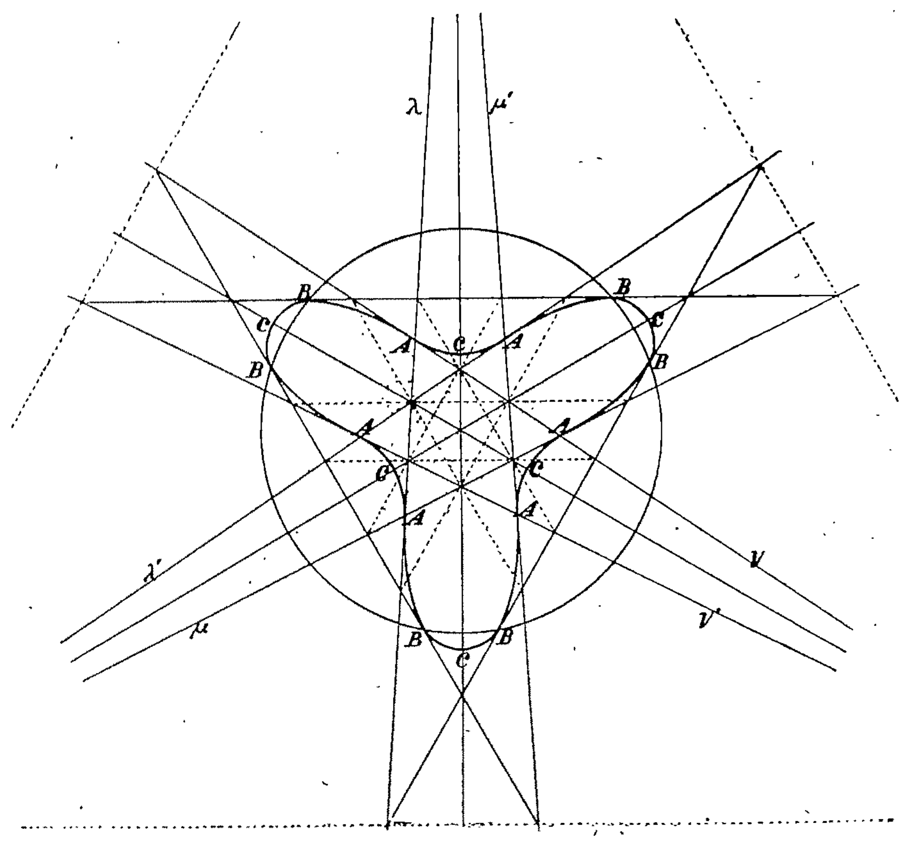

FIG. 4.

Hierbei sind die merkwürdigen Punkte alle mit den zugehörigen Buchstaben a, b, c bezeichnet, und haben offenbar die angegebene Reihenfolge. Die Figur ist ferner symmetrisch in Bezug auf drei Geraden durch den Mittelpunkt der Figur. Diese drei Symmetrieaxen sind als Perspectivitätsaxen aufzufassen, wobei die zugehörigen Perspectivitätscentra in einer Richtung die gegen die bezügliche Perspectivitätsaxe senkrecht steht im Unendlichen liegen. Die reellen Punkte c liegen zu je 2 auf diesen 3 Geraden, während die Punkte a auf einem Kreis, die Punkte b auf einem zweiten Kreis, um den Mittelpunkt der Figur liegen. Endlich bilden die Ecken des Coordinatendreiecks ein solches Wendetripel, wie wir gegen Ende des vorigen Paragraphen schon definirt haben, und vermöge der Symmetrie der Figur bilden die übrigen Punkte a ebenfalls ein solches Wendetripel. Ich werde in den folgenden Paragraphen zunächst angeben, wie diese Figur construirt worden ist.

§6.

Die reellen Punkte a und b.

I. *Die Punkte a.* Drei reelle Punkte a folgen sofort aus der Form der Curvengleichung. Denn, setzen wir da $\lambda = 0$ hinein, so folgt $\mu^3\nu = 0$. Die Gerade $\lambda = 0$ schneidet die Curve also wenn $\nu = 0$ einfach, wenn $\mu = 0$ dreifach. Es ist also $\lambda = 0$ eine Wendetangente im Punkte $\mu = 0$. Ebenfalls sind auch die Geraden $\mu = 0$ und $\nu = 0$ Wendetangenten im den Punkten $\nu = 0$, resp. $\lambda = 0$. Es ist also die Behauptung, dass die Ecken des Coordinatendreiecks ein Wendetripel bilden, bestätigt. Wir haben somit für diese 3 Wendepunkte resp. folgende Coordinaten:

$$0:0:1,\quad 1:0:0,\quad 0:1:0.$$

Um die drei anderen reellen Wendepunkte zu finden, ziehe ich die Untergruppe von sechs Collineationen heran, durch welche der reelle Curvenzug in sich übergeht. Diese Collineationen transformiren reelle Punkte in reelle Punkte und sind daher reell. Sie entsprechen genau den Substitutionen der G_6, bei der die Grundlinie des Fundamentalpolygons in sich übergeht. Die Collineationen von der Periode 3 lassen sich sofort aus der Form der Gleichung erschliessen. Die eine lautet nämlich:

$$U:^{*}\quad \lambda' = \mu,\quad \mu' = \nu,\quad \nu' = \lambda,$$

* Dass diese Collineation sich gerade so aus den erzeugenden Collineationen S und T zusammensetzen lässt, wie die entsprechende Substitution des Fundamentalpolygons, ist durch einfache Rechnung zu beweisen.

und die andere ist die Wiederholung von dieser. Die drei reellen Collineationen von der Periode 2 stellen einfach die Spiegelungen der Curve gegen die drei schon genannten Symmetrieaxen dar. Von diesen ist die eine diejenige Collineation, die wir früher mit T bezeichnet haben:

$$\begin{aligned}\lambda' &= A\lambda + B\mu + C\nu, \\ \mu' &= B\lambda + C\mu + A\nu, \\ \nu' &= C\lambda + A\mu + B\nu.\end{aligned}$$

Bei den Substitutionen der Periode 3 permutiren sich die schon gefundenen Wendepunkte cyclisch. Bei den perspectivischen Collineationen dagegen gehen dieselben in die drei Punkte über:

$$C:A:B, \quad A:B:C, \quad B:C:A.$$

Diese drei Punkte sind also die anderen reellen Wendepunkte der Curve, und bilden untereinander auch ein Wendetripel.

Aus diesen Collineationen erfahren wir nebenbei noch die Gleichungen der Perspectivitätsaxen. Denn dieselben teilen offenbar die inneren Winkel je zweier symmetrisch liegender Wendetangenten in zwei gleiche Hälften, und laufen dabei durch den Mittelpunkt der Figur $(1:1:1)$. Es lautet daher die Gleichung der einen Perspectivitätsaxe:

$$\lambda' + \lambda = \mu' + \mu = \nu' + \nu = 0,$$

die sich auch auf die Form bringen lässt:

$$BC\lambda + AB\mu + CA\nu = 0.$$

Die Gleichungen der beiden anderen Perspectivitätsaxen ergeben sich hieraus sofort durch cyclische Vertauschung der Coefficienten von λ, μ, ν.

II. *Die Punkte b.* Die Berechnung der reellen Doppeltangenten, bezw. ihrer Berührungspunkte geschieht am einfachsten, wenn man zunächst die folgende Identität betrachtet:

$$\begin{aligned}&\lambda^3\mu + \mu^3\nu + \nu^3\lambda \\ &\equiv 49\,(\lambda + \mu + \nu)(C^2\lambda + B^2\mu + A^2\nu)(B^2\lambda + A^2\mu + C^2\nu)(A^2\lambda + C^2\mu + B^2\nu) \\ &\qquad\qquad - [\lambda^2 + \mu^2 + \nu^2 + 3\,(\lambda\mu + \mu\nu + \nu\lambda)]^2 = 0.\end{aligned}$$

Aus dieser Form der Gleichung geht ohne Weiteres hervor, dass die reellen Geraden:

$$\begin{aligned}\lambda + \mu \quad + \nu \quad &= 0, \\ C^2\lambda + B^2\mu + A^2\nu &= 0, \\ B^2\lambda + A^2\mu + C^2\nu &= 0, \\ A^2\lambda + C^2\mu + B^2\nu &= 0,\end{aligned}$$

alle Doppeltangenten der Curve sind, und zwar liegen ihre Berührungspunkte sämmtlich auf dem Kegelschnitt (der sich vermöge der Symmetrie des Coordinatensystems als ein Kreis erweist):

$$\lambda^2 + \mu^2 + \nu^2 + 3\,(\lambda\mu + \mu\nu + \nu\lambda) = 0\,.$$

Die erste dieser Doppeltangenten ist die unendlich ferne Gerade, und ihre Berührungspunkte sind die imaginären Kreispunkte:

$$1 : \alpha : \alpha^2 \text{ und } 1 : \alpha^2 : \alpha\,.$$

Die übrigen 3 Doppeltangenten haben aber reelle Berührungspunkte, und diese sechs Punkte sind die gesuchten sechs reellen Punkte b.

§7.

Berechnung der Punkte c und einzelner intermediärer Punkte.

Wir haben oben schon bemerkt, dass die Punkte c auf der Curve von den 21 Perspectivitätsaxen ausgeschnitten werden. Da die Elimination zwischen der Gleichung einer Axe und der Curvengleichung zu einer Gleichung vierten Grades führt, deren Auflösung beträchtliche Schwierigkeiten darbieten würde, wenn man ohne besonderen Kunstgriff verführe, wird folgendermassen verfahren. Bei einer Collineation der Periode 4 bleiben drei Punkte der Ebene fest. Die Wiederholung einer solchen Collineation ergibt eine Collineation der Periode 2, also eine Perspectivität, bei welcher ein Punkt, das Perspectivitätscentrum, festbleibt, und eine gerade Linie, die Perspectivitätsaxe, Punkt für Punkt ungeändert bleibt. Die Collineation der Periode 4 ist daher für die Punkte der Perspectivitätsaxe nur von der Periode 2. Von den drei Fixpunkten dieser Collineation fällt dabei der eine notwendig mit dem betreffenden Perspectivitätscentrum zusammen, während die beiden anderen auf der zugehörigen Perspectivitätsaxe liegen. Die Punkte der Axe liegen also paarweise involutorisch, und die beiden Fixpunkte sind die Doppelpunkte dieser Involution. Wir wissen nun, dass irgend zwei zusammengehörige Punkte bei einer Involution zu den beiden Doppelpunkten derselben harmonisch liegen. Aber die Punkte c, die eine Axe aus der Curve ausschneidet, gehen bei der Collineation von der Periode 4 notwendig in sich über. Sie verteilen sich also auf zwei zu den auf der Perspectivitätsaxe gelegenen Fixpunkten harmonische Paare. Wir sind hiermit auf folgende Art der Berechnung geführt.

Es werden zuerst die beiden Fixpunkte auf der Perspectivitätsaxe aus den Transformationsformeln berechnet, was mit Hilfe einer Quadratwurzel geschehen wird, und dann erst mit deren Zuhilfenahme die Schnittpunkte der Axe mit der Curve vierter Ordnung. Bei letzterem Ansatze werden wir dabei notwendigerweise auf eine Gleichung vierten Grades geführt, in der die Coefficienten der ersten und dritten Potenzen der Veränderlichen Null sind, deren Auflösung also nur zwei Quadratwurzeln erfordert.

Ich werde dies jetzt ausführen. Zu dem Zwecke wähle ich die perspectivische Collineation T aus, und berechne die zugehörige Collineation der Periode 4, indem ich die entsprechenden Congruenzen der τ-Ebene zuhilfe nehme, und die Bildung derselben aus den erzeugenden Congruenzen untersuche. Es ergibt sich dass die Congruenz

$$R: \tau' \equiv \frac{2\tau + 2}{-2\tau + 2} \pmod 7$$

eine solche Congruenz von der Periode 4 ist, deren Wiederholung die erzeugende Congruenz:

$$T: \tau' \equiv -\frac{1}{\tau} \pmod 7$$

ergibt. Es kommt jetzt darauf an, R als Combination von S und T darzustellen. Es ergibt sich leicht:

$$R = S^3 T S^5 T S^3.$$

Dementsprechend lässt sich die gesuchte Collineation der Curve durch die entsprechende Combination der Collineationen S und T erzeugen. Ich unterdrücke die Zwischenrechnung, da sie durchaus einfacher Natur ist. Das Resultat ist, dass sich die Collineation R folgendermassen darstellt:

$$R: \begin{cases} \rho\lambda' = \gamma^6 (C\gamma^6\lambda + A\gamma^3\mu + B\gamma^5\nu), \\ \rho\mu' = \gamma^3 (A\gamma^6\lambda + B\gamma^3\mu + C\gamma^5\nu), \\ \rho\nu' = \gamma^5 (B\gamma^6\lambda + C\gamma^3\mu + A\gamma^5\nu). \end{cases}$$

Zur Kontrolle bilden wir uns die Wiederholung dieser Collineation:

$$\begin{aligned} \rho\lambda'' &= (C^2\gamma^3 + A^2\gamma^4 + B^2\gamma)\lambda + (CA + AB\gamma + BC\gamma^5)\mu + (BC\gamma^2 + CA\gamma^3 + AB)\nu \\ &= A\lambda + B\mu + C\nu, \\ \rho\mu'' &= (CA + AB\gamma + BC\gamma^5)\lambda + (A^2\gamma^4 + B^2\gamma^5 + C^2\gamma^2)\mu + (AB\gamma^6 + BC + CA\gamma^4)\nu \\ &= B\lambda + C\mu + A\nu, \\ \rho\nu'' &= (BC\gamma^2 + CA\gamma^3 + AB)\lambda + (AB\gamma^6 + BC + CA\gamma^4)\mu + (B^2\gamma + C^2\gamma^2 + A^2\gamma^6)\nu \\ &= C\lambda + A\mu + B\nu. \end{aligned}$$

Es ist also in der That die Collineation R eine solche, deren Wiederholung die Collineation T ergibt, d. h., die Collineation R ist eine derjenigen zwei, die wir suchen.

Setzen wir hier $\lambda' = \lambda$, $\mu' = \mu$, $\nu' = \nu$, so bekommen wir für ρ die drei Werte:

$$\rho = 1, + \sqrt{-1}, - \sqrt{-1};$$

und die Fixpunkte der Collineation R sind demnach das Perspectivitätscentrum:

$$\lambda : \mu : \nu = BC : AB : CA,$$

und die beiden auf der entsprechenden Perspectivitätsaxe liegenden Punkte:

$$1 + \sqrt{7} . B^2 : 1 + \sqrt{7} . A^2 : 1 + \sqrt{7} . C^2,$$
$$1 - \sqrt{7} . B^2 : 1 - \sqrt{7} . A^2 : 1 - \sqrt{7} . C^2.$$

Um nun die Schnittpunkte dieser Perspectivitätsaxe mit der Curve zu berechnen, setzen wir nach den Vorschriften der analytischen Geometrie

$$\lambda = (1 + \sqrt{7} . B^2) + \rho (1 - \sqrt{7} . B^2),$$
$$\mu = (1 + \sqrt{7} . A^2) + \rho (1 - \sqrt{7} . A^2),$$
$$\nu = (1 + \sqrt{7} . C^2) + \rho (1 - \sqrt{7} . C^2),$$

wo ρ ein Proportionalitätsfactor ist, in die Gleichung der Curve hinein. Wir bekommen so eine Gleichung vierten Grades in ρ, die sich auf folgende Form reducirt:

$$(21 - 8\sqrt{7}) \rho^4 + 6\rho^2 + (21 + 8\sqrt{7}) = 0,$$

wo also in der That die Glieder mit ρ^3 und ρ fehlen. Hieraus ergeben sich dann mit Hülfe von nur 2 Quadratwurzeln folgende Werte von ρ:

$$\rho = \pm \sqrt{21 + 8\sqrt{7}} \text{ und } \rho = \frac{\pm 1}{\sqrt{21 - 8\sqrt{7}}},$$

von denen die beiden ersten Werte reell, die beiden letzten dagegen imaginär sind.

Damit haben wir die zwei reellen Punkte c gefunden, die auf der gewählten Axe liegen. Hieraus folgen aber sofort alle sechs reelle c, denn es liegen auf jeder der reellen Axen zwei. Die Coordinaten der ersten zwei reellen c, die auf der Axe:

$$BC\lambda + AB\mu + CA\nu = 0$$

liegen, werden durch die Formeln gegeben:

$$\rho\lambda=(1+\sqrt{7}.B^2)\pm\sqrt{21+8\sqrt{7}}.(1-\sqrt{7}.B^2),$$
$$\rho\mu=(1+\sqrt{7}.A^2)\pm\sqrt{21+8\sqrt{7}}.(1-\sqrt{7}.A^2),$$
$$\rho\nu=(1+\sqrt{7}.C^2)\pm\sqrt{21+8\sqrt{7}}.(1-\sqrt{7}.C^2),$$

oder, numerisch: $\lambda:\mu:\nu=\ \ 2.24:-0.15:0.91$

und $\lambda:\mu:\nu=-3.01:4.73:1.28,$

und die Coordinaten der entsprechenden Punkte c auf den anderen Axen ergeben sich durch cyclische Vertauschung der hiermit gegebenen.

Die Coordinaten dieser Punkte a, b, c können wir natürlich auch aus den Reihen in τ berechnen. Indessen habe ich der algebraischen Methode den Vorzug gegeben, da die Beziehung zur Gruppe von 168 Transformationen dabei deutlicher hervortritt. Die τ-Reihen habe ich noch benutzt, um neben den jetzt gefundenen a, b, c einige intermediäre Punkte des reellen Curvenzugs zu berechnen. Ich gebe eine kleine Tabelle solcher Punkte, und bemerke nur noch, dass jeder Punkt, dessen Coordinaten in der Tabelle angegeben worden sind, eigentlich vermöge der dreifachen Symmetrie der Figur als Repräsentant eines Systems von sechs Punkten angesehen werden darf, insofern nur die geometrische Construction der Curve in Betracht kommt.

λ	μ	ν
— 0.003	0.273	2.730
— 0.06	0.71	2.35
— 0.28	1.09	2.18
— 0.36	1.19	2.17
— 0.92	1.67	2.26
— 1.59	2.31	2.38

Aus den in §§6, 7 nunmehr gefundenen Punkten ist Fig. 4 construirt worden.

§8.

Reelle Verbindungslinien imaginärer Punkte a.

Die übrigen Punkte der Curve sind nun alle imaginär; sie kommen aber immer paarweise conjugirt vor, wo jedes Paar durch eine reelle Gerade verbunden ist. Wir interessiren uns natürlich hauptsächlich für die Verbindungslinien, welche die imaginären Punkte a, b, c in diesem Sinne besitzen. Von denselben stehen einige in sehr einfacher Beziehung zu den schon bestimmten

Linien der Figur. Wir finden beispielsweise folgende Sätze über die Lage der Verbindungslinien imaginärer a.

Die 18 imaginären Punkte a ergeben sich aus den reellen a des zweiten Dreiecks durch Anwendung der Collineationen S, S^2, ... S^6. Nun gehen aus dem Punkte:

$$\lambda : \mu : \nu = A : B : C$$

bei den Substitutionen S, resp. S^6 folgende conjugirt imaginäre Punkte hervor:

$$\gamma A : \gamma^4 B : \gamma^2 C,$$
$$\gamma^6 A : \gamma^3 B : \gamma^5 C,$$

und die Gleichung der reellen Verbindungslinie derselben lautet:

$$BC\lambda + C^2\mu + B^2\nu = 0.$$

Die drei in Bezug auf das Coordinatendreieck symmetrisch liegenden Verbindungslinien, von denen die eine die eben angegebene ist, bilden ein Dreieck, dessen Ecken die Schnittpunkte folgender Paare von reellen Wendetangenten sind:

$$\begin{aligned} \lambda = 0,\ \lambda' &= A\lambda + B\mu + C\nu = 0, \\ \mu = 0,\ \nu' &= C\lambda + A\mu + B\nu = 0, \\ \nu = 0,\ \mu' &= B\lambda + C\mu + A\nu = 0. \end{aligned}$$

Ganz ähnliche Sätze gelten für die übrigen sechs Verbindungslinien conjugirt imaginärer Punkte a. Diese neun Verbindungslinien finden sich punktirt in der Figur vor.

Es besteht auch zwischen diesen Verbindungslinien und den reellen Doppeltangenten eine sehr einfache Relation. Es verbindet nämlich die Doppeltangente:

$$C^2\lambda + B^2\mu + A^2\nu = 0$$

die Schnittpunkte folgender sechs Paare von Geraden, von denen die einen reelle Wendetangenten, die anderen reelle Verbindungslinien imaginärer Wendepunkte sind:

$$\begin{aligned} \lambda = 0,\ & AB\lambda + B^2\mu + A^2\nu = 0, \\ \mu = 0,\ & C^2\lambda + CA\mu + A^2\nu = 0, \\ \nu = 0,\ & C^2\lambda + B^2\mu + BC\nu = 0, \\ \lambda' = 0,\ & BC\lambda + C^2\mu + B^2\nu = 0, \\ \mu' = 0,\ & A^2\lambda + AB\mu + B^2\nu = 0, \\ \nu' = 0,\ & A^2\lambda + C^2\mu + CA\nu = 0. \end{aligned}$$

Ganz ähnliche Sätze gelten auch für die beiden anderen im Endlichen gelegenen reellen Doppeltangenten, wie sofort an der Figur nachzuweisen ist.

§9.

Reciprocation der Figur.

Der Zielpunkt vorliegender Arbeit ist die Bildung der zu unserer Normalgleichung im Sinne der Einleitung gehörenden mehrfachen Ueberdeckung der Ebene. Hierbei darf man aber $\lambda:\mu:\nu$ nicht mehr als Punktcoordinaten betrachten, man muss sie vielmehr als Liniencoordinaten einführen. Um also die eigentliche Grundlage für diese Arbeit zu erhalten, müssen wir die Untersuchung der vorigen Paragraphen umkehren, und die Figur der reciproken Curve betrachten. Wir bekommen dabei, allgemein zu reden, eine Curve vierter Classe mit 24 Spitzen und 28 Doppelpunkten.

Speciell, die Reciprocation des reellen Curvenzugs ergibt einen reellen Curvenzug mit 6 reellen Spitzen und 3 reellen Doppelpunkten. Ausserdem besitzt die reciproke Curve, der isolirten unendlich fernen Doppeltangente entsprechend, welche die Curve vierter Ordnung hatte, ihrerseits einen reellen isolirten Doppelpunkt, der notwendigerweise im Mittelpunkte der Figur liegt.

Um dies jetzt constructiv durchzuführen, wird es offenbar vorteilhaft sein, in Bezug auf einen solchen Kegelschnitt zu reciprociren, dass die Coordinaten eines Punktes der einen Curve unter Festhaltung des Coordinatensystems unverändert die Coordinaten der entsprechenden Tangenten zur reciproken Curve wiedergeben. Eine solche Reciprocation wird mittelst des Kegelschnitts

$$\lambda^2 + \mu^2 + \nu^2 = 0$$

geleistet, denn die Gleichung der Polargeraden eines Punktes $\lambda_1:\mu_1:\nu_1$ lautet:

$$\lambda\lambda_1 + \mu\mu_1 + \nu\nu_1 = 0,$$

hat also dieselben Grössen λ_1, μ_1, ν_1 zu Coordinaten wie der Punkt selbst. Dieser Kegelschnitt ist aber, wenn wir dasselbe Coordinatensystem benutzen wie vorhin, ein imaginärer Kreis, und wir müssen zunächst die geometrische Deutung einer auf diesen imaginären Kreis bezüglichen Reciprocation untersuchen.

Will man in Bezug auf einen imaginären Kreis reciprociren, dessen Gleichung in gewöhnlichen rechtwinkligen Coordinaten lautet:

$$x^2 + y^2 + t^2 = 0,$$

so ist das dasselbe, als wenn man in Bezug auf den reellen Kreis

$$x^2 + y^2 - t^2 = 0$$

reciprocirt, und dann die ganze Figur um 180° um den Mittelpunkt herumdreht. Um das entsprechende Resultat für unseren Fall durchzuführen, setzen wir:

$$\begin{aligned} \lambda + \mu \ + \nu \ &= t, \\ \lambda + \alpha\mu + \alpha^2\nu &= x + yi, \\ \lambda + \alpha^2\mu + \alpha\nu &= x - yi, \end{aligned}$$

wo

$$\alpha = e^{\frac{2i\pi}{3}}.$$

Diese Transformation entspricht einer Transformation auf ein rechtwinkliges Coordinatensystem (x, y), wo die x-Axe parallel zur Geraden $\lambda = 0$ durch den Mittelpunkt des Coordinatendreiecks gezogen ist, und die y-Axe senkrecht zur x-Axe steht und ebenfalls durch den Mittelpunkt des Coordinatensystems läuft, wie in folgender Figur gezeigt wird:

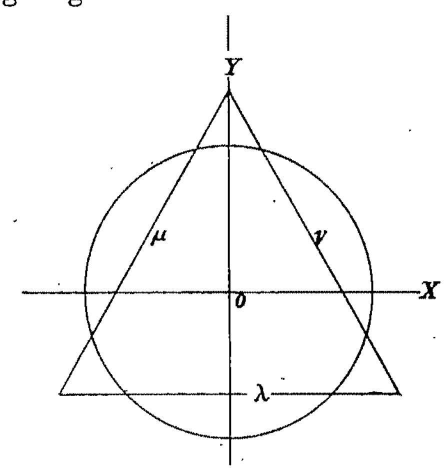

FIG. 5.

Aus den Transformationsformeln haben wir sofort:

$$\begin{aligned} \lambda^2 + \mu^2 + \nu^2 + 2(\lambda\mu + \mu\nu + \nu\lambda) &= t^2, \\ \lambda^2 + \mu^2 + \nu^2 - \ (\lambda\mu + \mu\nu + \nu\lambda) &= x^2 + y^2. \end{aligned}$$

Also lautet die Gleichung des imaginären Kreises in rechtwinkligen Coordinaten:

$$2(x^2 + y^2) + t^2 = 0,$$

und die Gleichung des entsprechenden reellen Kreises:

$$2(x^2 + y^2) - t^2 = 0$$

lautet in homogenen Coordinaten λ, μ, ν:

$$\lambda^2 + \mu^2 + \nu^2 - 4(\lambda\mu + \mu\nu + \nu\lambda) = 0.$$

Wir können also die Reciprocation auf die Weise geometrisch durchführen, dass wir in Bezug auf den letztgenannten Kreis (der übrigens in der Figur ganz genau gezeichnet ist) reciprociren, und dann die ganze Figur um 180° um den Mittelpunkt herumdrehen.

Es führt indessen zu einer genaueren Zeichnung, wenn wir nicht die neue Figur in der hiermit geschilderten Weise aus der ursprünglichen Figur geometrisch herleiten, sondern die Punktcoordinaten der hauptsächlichen Punkte derselben algebraisch berechnen und darnach die Figur unabhängig construiren. Wir führen also die Construction der Curve vierter Classe in der Weise aus, dass wir erst die Gleichungen der Tangenten in den einzelnen von uns festgelegten Punkten der Curve vierter Ordnung berechnen; wir haben dann in den Coefficienten einer solchen Gleichung die Coordinaten des entsprechenden Punktes der Curve vierter Classe, und tragen diese in die Figur ein. Die Curve hat dann diefolgende Gestalt:

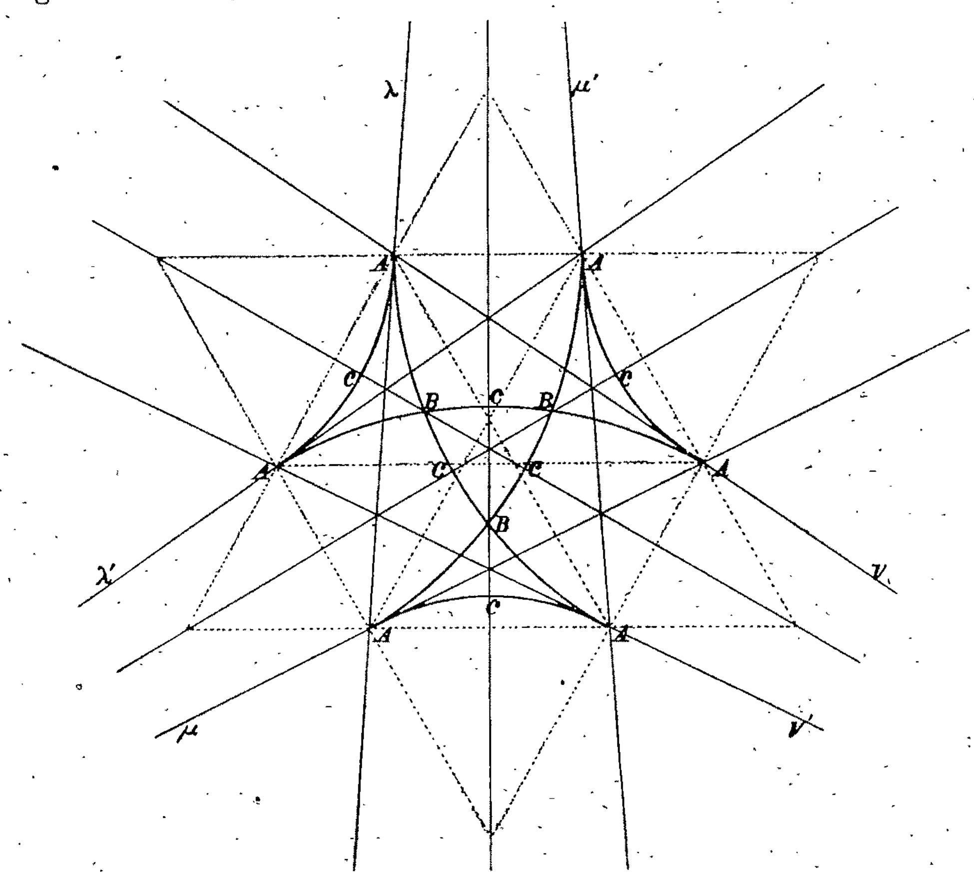

FIG. 6.

Die genauere Ausführung dieser Figur wird sofort angegeben. Wir bemerken zunächst, dass diese Curve auch wieder in Bezug auf dieselben drei Symmetrieaxen symmetrisch liegt, wie die ursprüngliche Curve vierter Ordnung; es war dies von vornherein zu erwarten.

§10.

Die singulären Punkte der reciproken Curve.

Ich gehe jetzt zur Berechnung der merkwürdigen Stellen der Curve vierter Classe über. Hierbei werde ich nach wie vor immer von den "Punkten" a, b, c sprechen, indem ich so die *Stellen* bezeichne, die den früheren a, b, c correspondiren, und die also den *Tangenten* entsprechen, welche die Curve vierter Ordnung in den Punkten a, b, c besass. Dieselben erweisen sich von principieller Wichtigkeit für die zu bildende Ueberdeckung der Ebene.

I. *Die* 6 *reellen Punkte* a. Es hat sich bei der Curve vierter Ordnung ergeben, dass die 6 reellen Wendetangenten zwei Dreiecke bilden, deren Ecken gerade die Wendepunkte sind. Dem genau entsprechend, bilden hier bei der Curve vierter Classe die 6 reellen Spitzentangenten zwei Dreiecke, deren Ecken die Spitzen sind. Von diesen Dreiecken ist das eine, wie sofort aus den entsprechenden Formeln für die Curve vierter Ordnung zu ersehen ist, das Coordinatendreieck (λ, μ, ν) selbst. Das zweite Dreieck wird aus denselben Gründen eben von denselben Geraden gebildet, wie das entsprechende Dreieck (λ', μ', ν') der Curve vierter Ordnung. Die Coordinaten der Spitzen a sind also identisch mit den Coordinaten der Wendepunkte, und somit bezw. durch die Formeln gegeben:

$$0:0:1\ ,\quad 1:0:0\ ,\quad 0:1:0\ ,$$
$$C:A:B,\quad A:B:C,\quad B:C:A.$$

II. *Die reellen Punkte* b. Es gibt, wie oben schon bemerkt, vier reelle Doppelpunkte, von denen der eine im Mittelpunkt der Figur isolirt liegt. Dieselben vereinigen in sich je 2 Stellen b, je nachdem man nämlich den einen oder den anderen Ast der Curve, der durch sie bezw. hindurchgeht, meint, und ihre Coordinaten ergeben sich sofort aus den Gleichungen der entsprechenden Doppeltangenten der Curve vierter Ordnung:

$$1:1:1 \text{ (der isolirte Punkt)},$$
$$C^2:B^2:A^2,\quad B^2:A^2:C^2,\quad A^2:C^2:B^2.$$

III. *Die 6 reellen Punkte c.* Da die algebraischen Formeln für die sextaktischen Punkte sehr complicirt sind, begnüge ich mich damit, dass ich die Coordinaten derselben numerisch angebe. Auf der Symmetrieaxe, deren Gleichung lautet:

$$BC\lambda + AB\mu + CA\nu = 0$$

liegen die beiden Punkte c:

$$-0.30 : 2.20 : 1.09$$

und

$$1.40 : 0.63 : 0.98,$$

während die entsprechenden Punkte auf den anderen beiden Symmetrieaxen sich durch cyclische Vertauschung dieser Coordinaten ergeben.

IV. *Schnittpunkte imaginärer Spitzentangenten.* Es geht bei der Reciprocation jedes Paar conjugirt imaginärer Punkte der Curve vierter Ordnung mit reeller Verbindungslinie in zwei conjugirt imaginäre Tangenten der Curve vierter Classe mit reellem Schnittpunkt über. Was die Lage der Schnittpunkte der imaginären Spitzentangenten angeht, so haben wir, dem §8 dualistisch entsprechend, folgende Sätze:

1. Die drei Schnittpunkte dieser Art, deren Coordinaten bezw. sind:

$$BC : C^2 : B^2, \quad C^2 : B^2 : BC, \quad B^2 : BC : C^2$$

sind die Ecken des aus den Verbindungslinien folgender Paare von reellen Punkte a gebildeten Dreiecks:

$$1:0:0, \quad A:B:C,$$
$$0:1:0, \quad C:A:B,$$
$$0:0:1, \quad B:C:A.$$

Für die beiden anderen Tripel von Schnittpunkten conjugirt imaginärer Spitzentangenten gelten ähnliche Sätze, wie aus der Figur zu ersehen, wo alle genannten Punkte und Linien eingetragen sind.

2. In dem Doppelpunkte $\quad C^2 : A^2 : B^2$

stossen die Verbindungslinien der folgenden sechs Punktepaare zusammen, wo jedesmal der eine Punkt eines Paares eine reelle Spitze ist, der zweite Punkt der reelle Schnittpunkt zweier conjugirt imaginärer Spitzentangenten:

$$1 : 0 : 0, \quad AB : B^2 : A^2,$$
$$0 : 1 : 0, \quad C^2 : CA : A^2,$$
$$0 : 0 : 1, \quad C^2 : B^2 : BC,$$
$$A : B : C, \quad BC : C^2 : B^2,$$
$$B : C : A, \quad A^2 : AB : B^2,$$
$$C : A : B, \quad A^2 : C^2 : CA.$$

Dritter Abschnitt.

Einführung und Haupteigenschaften der zu der Curve vierter Classe gehörenden mehrfachen Ueberdeckung der Ebene. Aufgabenstellung.

An die nunmehr gezeichnete Curve vierter Classe schliesst sich die schon besprochene mehrfache Ueberdeckung der Ebene an, deren Gesammtverlauf jetzt untersucht werden soll.

§11.

Die Ueberdeckungen verschiedener Teile der Ebene.

Die Frage nach der Anzahl der Blätter, mit welchen jeder Teil der Ebene überdeckt zu denken ist, ist identisch mit der Frage: wie viele von den Tangenten an die Curve von jedem Punkte des betreffenden Teils der Ebene aus sind imaginär? Dies lässt sich durch folgende einfache Betrachtung sofort erschliessen. Von jedem Punkte der Ebene aus laufen an die Curve vierter Classe, algebraisch zu reden, vier (reelle oder imaginäre) Tangenten. Für einen gewöhnlichen Punkt auf der reellen Curve fallen von diesen Tangenten zwei zusammen. Nehmen wir nun zwei benachbarte Stellen auf der einen, resp. auf der anderen Seite der Curve. An Stelle der zusammenfallenden Tangenten des Curvenpunktes treten an der convexen Seite der Curve zwei reelle und verschiedene Tangenten ein, an der concaven Seite dagegen zwei imaginäre Tangenten. Das Verhalten der übrigen Tangenten, wenn wir von dem einen Punkte zum anderen gehen, bleibt ungeändert. Wir haben somit folgende

Regel: *Beim Uebergang von der convexen zur concaven Seite des reellen Curvenzugs wird die Anzahl der imaginären Tangenten von einem Punkte aus um zwei vermehrt.*

Die Anzahl der imaginären Tangenten von jedem Punkte der einzelnen Gebiete oder, was auf dasselbe hinauskommt, die Anzahl der Blätter der mehrfachen Ueberdeckung, wird dann durch die Ziffern der folgenden Figur angegeben. Denn es laufen nach der gegebenen Regel von den Punkten des äusseren Gebietes zwei imaginäre Tangenten weniger an die Curve als von den Punkten des inneren Gebietes, in welchem der isolirte Doppelpunkt sich befindet, und zwei imaginäre Tangenten mehr als von den Punkten der übrigen drei dreieckförmigen Gebiete, und die Anzahl solcher imaginärer Tangenten beträgt höch-

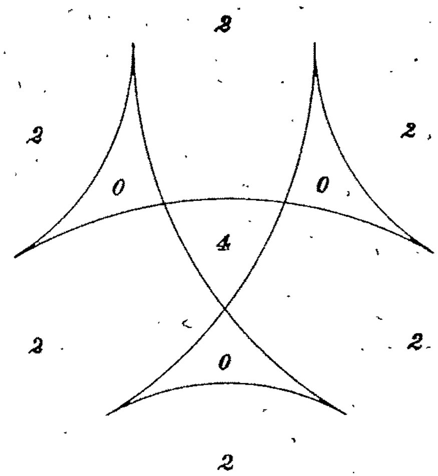

FIG. 7.

stens vier. Es lässt sich auch leicht unmittelbar sehen, dass von jedem Punkte eines der dreieckförmigen Gebiete in der That vier reelle Tangenten an die Curve laufen.

§12.

Von dem Zusammenhange der verschiedenen Ueberdeckungen.

Jetzt kommt der Gedanke zur Geltung, den wir schon in der Einleitung besprochen haben. Wir ordnen jede Gerade unseres algebraischen Gebildes in der Weise einem Punkte zu, dass jede imaginäre Tangente ihrem einen reellen Punkt und dementsprechend jede reelle Tangente ihrem Berührungspunkt entspricht. Damit diese Beziehung ein-eindeutig wird, denken wir uns die Ebene in jedem Teile aus so vielen Blättern bestehend, wie die Anzahl der von einem Punkte desselben zu vertretenden Tangenten beträgt, wobei der Punkt in dem einen oder in dem anderen Blatte gelegen zu denken ist, je nachdem die eine oder die andere Tangente in Betracht zu ziehen ist. Diese Anzahl ist nun gerade in der obigen Figur angegeben.

Diese Blätter sind dabei insoweit paarweise zusammengeordnet, als die Tangenten eines jeden Punktes paarweise conjugirt imaginär vorkommen, und wir werden von dem oberen, bezw. dem unteren Blatte jedes solchen Paares reden dürfen. In jedem Blatte denken wir uns nun die Flächennormale nach aussen hin gerichtet, also in einem oberen Blatte nach oben, in einem unteren Blatte nach unten. Demnach sprechen wir von der positiven oder von der negativen

Seite eines Blattes, je nachdem die Flächennormale nach der betreffenden Seite hin gerichtet ist, oder nicht. Was den Zusammenhang der verschiedenen Blätter angeht, können wir folgende Angaben machen, deren nähere Untersuchung, resp. Beweis bis auf ein weiteres Kapitel verschoben wird.

Im Unendlichen zeigen sich keine wesentlichen Singularitäten. Es gehen demnach die beiden Blätter ohne Verzweigung in einander über, das obere Blatt in das untere, und umgekehrt,—und zwar die positive Seite des oberen Blattes in die positive Seite des unteren Blattes, die negative Seite in die negative. Dies stimmt mit unserer Angabe des Sinnes der Flächennormale, denn dieselbe dreht sich, den Vorstellungsweisen der projectiven Geometrie entsprechend, um 180°, wenn man durch das Unendliche hindurchläuft, und ein Punkt auf der positiven Seite des einen Blattes muss dabei ins andere Blatt geraten, damit er auf der positiven Seite bleibt.

Längs des reellen Curvenzugs sind jedesmal solche zwei Blätter zusammengebunden zu denken, welche dort überhaupt aufzuhören scheinen, wobei wieder die positive Seite des einen Blattes in die positive Seite des anderen Blattes übergehen muss und umgekehrt. Es hängen also die zwei Blätter, welche nur im innersten Gebiete existiren, längs ihrer ganzen, von drei krummlinigen Stücken gebildeten Contour mittelst einer Falte zusammen, während die beiden anderen Blätter die ganze Ebene mit Ausnahme von den drei äusseren dreieckförmigen Gebieten überdecken und längs der Contour der letzteren mittelst einer Falte zusammenhängen.

Zu dem isolirten Doppelpunkt gehören zwei Paare zusammenfallender Tangenten. Dieser Punkt zählt also als gemeinschaftlicher Punkt der beiden oberen Blätter, bezw. der beiden unteren. Es zeigt sich ferner, dass man erst nach zweimaligem Umlauf um diesen Punkt herum nach einem gegebenen Punkt seiner Umgebung wieder zurückgelangt. *Der isolirte Doppelpunkt ist also ein Doppelverzweigungspunkt der Ueberdeckung*, d. h., ein einfacher Verzweigungspunkt sowohl für die beiden oberen Blätter, wie für die beiden unteren. Es besteht also um den Mittelpunkt herum ein Zusammenhang zwischen jedem der beiden grossen Blätter und dem zugehörigen kleinen Blatte. Um daher von einem grossen Blatte in das zugehörige kleine Blatt zu gelangen, und umgekehrt, müssen wir uns diese beiden Verzweigungspunkte durch einen Verzweigungschnitt, oder auch durch irgend eine ungerade Zahl von Verzweigungsschnitten verbunden denken. Man weiss von der gewöhnlichen Riemann'schen Fläche, dass diese Verzweigungs-

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schnitte in hohem Masse willkürlich sind. Der Symmetrie halber ziehe ich drei solche Verzweigungsschnitte, und zwar in der Figur geradlinig, indem ich jedesmal durch einen der Doppelpunkte des reellen Curvenzugs hindurchgehe. Ein solcher Verzweigungsschnitt erstreckt sich dann vom Mittelpunkte aus zunächst in den beiden oberen Blättern verlaufend bis zu einem der drei Doppelpunkte hin, um dann von diesem in den beiden unteren Blättern verlaufend wieder zum Mittelpunkte zurückzukehren.

Wir haben nun endlich in der folgenden Figur ein Bild der Ueberdeckung, wobei der Zusammenhang der *beiden oberen Blätter* durch Punktirung eines Weges angedeutet ist, und das grosse Blatt als über dem kleinen Blatt gelegen zu denken ist.

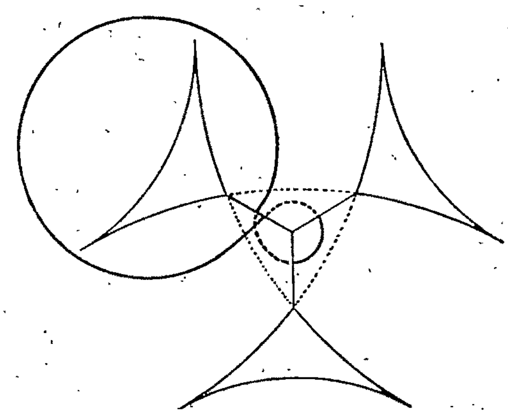


FIG. 8.

VIERTER ABSCHNITT.

Uebertragung der Symmetrielinien auf die mehrfache Ueberdeckung der Ebene.

Es ist nun die Aufgabe, die Beziehung der mehrfachen Ueberdeckung der Ebene zum Fundamentalpolygon so deutlich wie möglich hervortreten zu lassen. Hiernach werden wir vor allen Dingen verlangen, dass in diese Figur die den 28 Symmetrielinien entsprechenden Curven hineingetragen werden, wobei dieselbe in 2.168 Dreiecke zerlegt wird, welche einzeln den Dreiecken des Fundamentalpolygons genau entsprechen.

§13.

Allgemeiner Ansatz.

Es ist oben schon gezeigt worden, dass bei den 168 Transformationen des (geschlossenen) Fundamentalpolygons in sich jede der 28 Symmetrielinien stets wieder in eine solche übergeführt wird. Es is nun die Frage, wie wird dieses Vorkommen auf der mehrfachen Ueberdeckung der Ebene wiedergegeben? Um diese Frage zu beantworten, bemerken wir zunächst, wie wir schon hervorgehoben haben, dass der reelle Curvenzug der einen Symmetrielinie, die wir die Grundlinie genannt haben, entspricht. Es bleibt also übrig, die Natur der reellen Curven zu untersuchen, in die der reelle Curvenzug vermöge unserer Festsetzungen bei der Gruppe der 168 Collineationen transformirt wird. Bei den 6 reellen Collineationen zunächst geht derselbe einfach in sich über. Wendet man aber eine imaginäre Collineation auf ihn an, so gehen seine reellen Tangenten zunächst in ein Aggregat von imaginären Tangenten über, von welchen jede ihr Bild auf der mehrfachen Ueberdeckung der Ebene in ihrem einen reellen Punkte findet. Dieses Aggregat von Punkten stellt ersichtlich eine zweite Symmetrielinie des Fundamentalpolygons dar. Ich behaupte und werde sofort beweisen:

Jedes solche Aggregat von Punkten, welches auf der mehrfachen Ueberdeckung der Ebene einer Symmetrielinie des Fundamentalpolygons entspricht, bildet den reellen Zug einer algebraischen Curve; und zwar geht diese Curve aus der Grundcurve durch eine quadratische Transformation hervor.

Beweis: Es sei

$$\lambda u + \mu v + \nu w = 0$$

die Gleichung einer reellen Tangente der Curve vierter Classe. Wendet man irgend eine imaginäre Collineation Σ der Gruppe auf dieselbe an, so entsteht die Tangente:

$$\phi u + \psi v + \chi w = 0,$$

wo $\phi:\psi:\chi$ lineare Functionen von $\lambda:\mu:\nu$ sind. Dieselbe wird auf der mehrfachen Ueberdeckung der Ebene durch ihren einen reellen Punkt, den Schnittpunkt der conjugirt imaginären Tangente:

$$\phi' u + \psi' v + \chi' w = 0$$

vertreten. Die Coordinaten des betreffenden reellen Schnittpunktes sind demnach durch die Formeln gegeben:

$$Q:\quad \begin{aligned} \rho u &= \psi\chi' - \psi'\chi, \\ \rho v &= \chi\phi' - \chi'\phi, \\ \rho w &= \phi\psi' - \phi'\psi, \end{aligned}$$

und diese Formeln stellen offenbar eine in den λ, μ, ν quadratische Transformation dar. Eliminirt man nun die drei Grössen ρ, $\lambda:\mu:\nu$ zwischen diesen drei Gleichungen und der Curvengleichung

$$\lambda^3\mu+\mu^3\nu+\nu^3\lambda=0,$$

so bekommt man eine algebraische Gleichung in $u:v:w$, die Gleichung der Symmetrielinie, die wir suchen.

Dass die gesuchte Symmetrielinie im Allgemeinen gerade den reellen Zug der auf diese Weise gefundenen Curve bildet, ergibt sich sofort aus der Form der Transformationsgleichungen Q. Diese Gleichungen haben nämlich reelle Coefficienten und transformiren reelle $\lambda:\mu:\nu$ in reelle $u:v:w$, imaginäre $\lambda:\mu:\nu$ in imaginäre $u:v:w$. Die einzige Ausnahme bildet der in der That vorkommende Fall, dass zwischen den Transformationsgleichungen eine lineare, von $\lambda\cdot\mu:\nu$ unabhängige, algebraische Relation besteht. In diesem Falle geht bei der Transformation die ganze Ebene in eine gerade Linie über, und die gesuchte Symmetrielinie bildet nur einen Teil der ganzen Linie.

Was den Grad der so entstehenden Curven angeht, so beachte man folgendes. (Der Ausnahmefall wird einstweilen bei Seite gelassen.) Die drei Kegelschnitte

$$\begin{aligned} &\text{(I)} && \psi\chi' - \psi'\chi = 0, \\ &\text{(II)} && \chi\phi' - \chi'\phi = 0, \\ &\text{(III)} && \phi\psi' - \phi'\psi = 0 \end{aligned}$$

haben immer 3 Punkte gemeinsam. Denn aus den Gleichungen (I) und (II) folgt sofort (III), sofern man von dem einen Schnittpunkte von (I) und (II) absieht, welcher in $\chi=\chi'=0$ hineinfällt. Die Transformation Q ist also nach bekannten Principien der analytischen Geometrie immer eine eindeutig umkehrbare (eigentliche quadratische) Transformation. Die drei gemeinsamen Schnittpunkte der drei Kegelschnitte (I), (II) und (III) nennen wir die *Fundamentalpunkte* der Transformation.

Die Umkehrtransformation lautet dann etwa:

$$\begin{aligned} \sigma\lambda &= \Phi(u, v, w), \\ \sigma\mu &= \Psi(u, v, w), \\ \sigma\nu &= \mathrm{X}(u, v, w), \end{aligned}$$

wo Φ, Ψ, X Formen zweiter Ordnung sind. Setzt man diese Werte für λ, μ, ν in die Gleichung der Curve

$$\lambda^3\mu + \mu^3\nu + \nu^3\lambda = 0$$

hinein, so bekommt man eine Gleichung 8^{ten} Grades in $u:v:w$. Läuft aber die Grundcurve durch einen oder mehrere der Fundamentalpunkte, so treten in diese Gleichung 8^{ten} Grades, der bekannten Theorie der quadratischen Transformation zufolge, lineare Factoren ein, von denen man absehen kann. Denn in diesem Falle entspricht einem solchen Punkt $\lambda:\mu:\nu$ nicht ein bestimmter Punkt $u:v:w$, sondern eine ganze gerade Linie, die nichts mit der gesuchten Curve zu thun hat.* Es reducirt sich dann die gesuchte Gleichung auf eine Gleichung bezw. des siebenten, sechsten oder fünften Grades.

Um auf den oben erwähnten Ausnahmefall zurückzukommen, bemerken wir: in diesem Fall haben die drei Kegelschnitte vermöge der zwischen ihren Gleichungen bestehenden linearen Relation nicht nur drei, sondern sogar vier Schnittpunkte gemein, und bilden drei Curven eines Curvenbüschels durch die vier Punkte. Eine solche Transformation nennt man eine *uneigentliche* quadratische Transformation. Ihre Bedeutung ergibt sich später so einfach, dass von einer vorherigen allgemeinen Erläuterung abgesehen werden kann.

§14.

Reduction der Aufgabe.

Ehe wir jetzt gleich zur Berechnung der einzelnen Symmetrielinien übergehen, wird es vorteilhaft sein, zu untersuchen, in wie fern sich die Aufgabe vermöge der einfachen Bedeutung der reellen Collineationen vereinfachen lässt. Unter der Festsetzung eines gleichseitigen Dreiecks als Coordinatendreieck haben wir gesehen, dass die Operationen der G_6, welche den reellen Curvenzug in sich überführen, folgende Bedeutung haben. Die Transformationen der Periode 3 stellen eine Drehung um den Mittelpunkt der Figur herum dar. Die Transformationen der Periode 2 ergeben eine Spiegelung der Figur im gewöhnlichen Sinne in Bezug auf die drei Symmetrieaxen derselben. Dieselbe Bedeutung müssen diese Transformationen auch für die ganze Ueberdeckung der Ebene haben.

Jetzt ist es klar, warum wir schon im Fundamentalpolygon die Symmetrie-

* Salmon, Higher Plane Curves, Art. 358.

linien in Bezug auf die entsprechende Untergruppe G_6 in Classen zerlegt haben. Denn von jeder Gruppe von Curven, welche bei den Transformationen dieser G_6 in einander übergehen, brauchen wir jedesmal nur eine zu berücksichtigen, und ihre Gleichung zu berechnen. Die anderen ergeben sich dann sofort bei Anwendung der einfachen Transformationen der reellen G_6.

In Bezug auf die G_6 gab es nun vier verschiedene Classen von Symmetrielinien im Fundamentalpolygon der s-Ebene: (1) die drei Curven, welche sich selbst symmetrisch liegen, von denen jede durch zwei Punkte c der Grundlinie hindurchläuft; (2) die drei Paare symmetrisch liegender Linien, von denen jedes Paar durch zwei Punkte b der Grundlinie hindurchläuft; (3) die drei Paare symmetrisch liegender Curven, welche die Grundlinie gar nicht schneiden; (4) die sechs Paare symmetrisch liegender Curven, von welchen drei Paare durch drei zusammengehörige Punkte a der Grundlinie, die übrigen drei Paare durch die anderen drei Punkte a der Grundlinie hindurchlaufen.

Wir bemerken nun, dass symmetrisch liegende Linien des Fundamentalpolygons übereinanderliegende Curven der mehrfachen Ueberdeckung der Ebene ergeben. Denn symmetrisch liegende Punkte des Fundamentalpolygons ergeben conjugirt imaginäre Tangenten der Curve vierter Classe und werden daher durch übereinanderliegende Punkte der mehrfachen Ueberdeckung dargestellt. Jedes Paar von symmetrisch liegenden Linien des Fundamentalpolygons wird also durch eine und dieselbe Curve der uvw-Ebene repräsentirt. Wir bekommen also in der uvw-Ebene

3 Symmetrielinien der ersten Art,
3 Symmetrielinien der zweiten Art,
3 Symmetrielinien der dritten Art,
6 Symmetrielinien der vierten Art,

wobei wir jedesmal nur *eine, repräsentirende* Curve auswählen, und ihre Gleichung berechnen, indem die anderen aus ihr durch die 6 reellen Collineationen hervorgehen.

§15.

Die Symmetrielinien der ersten Art.

Von den Symmetrielinien der ersten Art greifen wir diejenige heraus, die wir schon genauer betrachtet haben, und die bei den conjugirten Transforma-

tionen S^4TS, resp. S^3TS^6 aus der Grundlinie hervorgeht. Die entsprechende Collineation S^4TS wird durch die Formeln gegeben:

$$\begin{aligned}\rho\lambda' &= \gamma\ (A\gamma^4\lambda + B\gamma^2\mu + C\gamma\nu),\\ \rho\mu' &= \gamma^4 (B\gamma^4\lambda + C\gamma^2\mu + A\gamma\nu),\\ \rho\nu' &= \gamma^2 (C\gamma^4\lambda + A\gamma^2\mu + B\gamma\nu)\end{aligned}$$

und die Collineation S^3TS^6 ist einfach die conjugirt imaginäre Collineation.

Für die gesuchte Curve lauten alsdann die Transformationsformeln:

$$\begin{aligned}\sigma u &= (\mu'\nu'') = A\left(\lambda^2BC + \mu^2CA + \nu^2AB + \mu\nu C^2 - \nu\lambda(B^2 + C^2) + \lambda\mu B^2\right),\\ \sigma v &= (\nu'\lambda'') = C\left(\lambda^2CA + \mu^2AB + \nu^2BC + \mu\nu A^2 + \nu\lambda B^2 - \lambda\mu(A^2 + B^2)\right),\\ \sigma w &= (\lambda'\mu'') = B\left(\lambda^2AB + \mu^2BC + \nu^2CA - \mu\nu(C^2 + A^2) + \nu\lambda C^2 + \lambda\mu A^2\right).\end{aligned}$$

Hier bemerkt man sofort, dass zwischen diesen Transformationsformeln eine von $\lambda:\mu:\nu$ unabhängige Relation besteht, nämlich:

$$\frac{u}{A} + \frac{v}{C} + \frac{w}{B} = 0$$

oder,

$$BCu + ABv + CAw = 0,$$

welches die Gleichung von einer der reellen Symmetrieaxen der Figur ist. Bei dieser Transformation geht also die ganze Ebene in diese eine Gerade über, und die von uns gesuchte Symmetrielinie bildet jedenfalls einen Teil der Geraden.

Die beiden anderen Symmetrielinien der ersten Art sind die entsprechenden Teile der beiden anderen reellen Symmetrieaxen. Ihr Verlauf auf der mehrfachen Ueberdeckung wird später (§20) gegeben.

§16.

Die Symmetrielinien der zweiten Art.

Von den Symmetrielinien der zweiten Art im Fundamentalpolygon schneidet jedes Paar symmetrisch liegender Linien die Grundcurve in zwei Punkten b. Diese Punkte b sind jedesmal diejenigen, deren Vertreter auf der mehrfachen Ueberdeckung zu einem Doppelpunkt der Curve vierter Classe sich vereinigen. Daraus schliessen wir sofort, dass von den betreffenden Symmetrielinien der zweiten Art auf der mehrfachen Ueberdeckung jede einen Doppelpunkt besitzt, der mit einem der reellen Doppelpunkte der Curve vierter Classe zusammenfällt. Als Repräsentanten dieser Art nehmen wir diejenige Curve, welche die beiden

symmetrisch liegenden Linien, die aus der Grundlinie bei den Transformationen S^3TS^3, bezw. S^4TS^4 hervorgehen, vertritt. Die entsprechende Collineation S^3TS^3 lautet

$$\begin{aligned}\rho\lambda' &= \gamma^3(A\gamma^3\lambda + B\gamma^5\mu + C\gamma^6\nu),\\ \rho\mu' &= \gamma^5(B\gamma^3\lambda + C\gamma^5\mu + A\gamma^6\nu),\\ \rho\nu' &= \gamma^6(C\gamma^3\lambda + A\gamma^5\mu + B\gamma^6\nu),\end{aligned}$$

und die Collineation S^4TS^4 ist einfach die conjugirt imaginäre Collineation. Bei Zusammensetzung dieser beiden Collineationen auf die vorgeschriebene Weise, erhält man folgende quadratische Transformation:

$$\begin{aligned}\sigma u &= \lambda^2BC^2 + \mu^2AC^2 + \nu^2ABC + \mu\nu ABC + 2\nu\lambda A^2B + 2\lambda\mu ABC,\\ \sigma v &= \lambda^2ABC + \mu^2AB^2 + \nu^2B^2C + 2\mu\nu ABC + \nu\lambda ABC + 2\lambda\mu C^2A,\\ \sigma w &= \lambda^2A^2B + \mu^2ABC + \nu^2A^2C + 2\mu\nu B^2C + 2\nu\lambda ABC + \lambda\mu ABC.\end{aligned}$$

Um die Fundamentalpunkte dieser Transformation zu berechnen, bemerke man zunächst, dass die beiden Collineationen S^3TS^3 und S^4TS^4 nicht nur conjugirt imaginäre Collineationen, sondern auch gerade die Wiederholungen von einander sind. Es ist

$$(S^3TS^3)^2 = S^4TS^4,$$

und umgekehrt

$$(S^4TS^4)^2 = S^3TS^3.$$

Es ist hiernach klar, dass alle drei Kegelschnitte: $u=0$, $v=0$ und $w=0$ durch die drei bei diesen Collineationen festbleibenden Punkte laufen müssen, und diese Punkte sind: erstens, der Punkt

$$\lambda:\mu:\nu = B^2:A^2:C^2,$$

welcher nicht auf der Grundcurve liegt, und dann die beiden Berührungspunkte der Doppeltangenten:

$$B^2\lambda + A^2\mu + C^2\nu = 0,$$

deren Coordinaten bezw. sind:

$$\lambda:\mu:\nu = \frac{1}{A(\gamma^3+\alpha\gamma^4)} : \frac{1}{C(\gamma^5+\alpha\gamma^2)} : \frac{1}{B(\gamma^6+\alpha\gamma)},$$

$$\lambda:\mu:\nu = \frac{1}{A(\gamma^3+\alpha^2\gamma^4)} : \frac{1}{C(\gamma^5+\alpha^2\gamma^2)} : \frac{1}{B(\gamma^6+\alpha^2\gamma)},$$

wo

$$\alpha = e^{\frac{2i\pi}{3}}.$$

Die transformirte Curve wird also von der sechsten Ordnung sein. Die Herstellung der Gleichung derselben bietet keine principiellen Schwierigkeiten, ist indessen so umständlich, dass ich mich damit begnüge, dass ich ein Bild der-

selben angebe, das sich vermittelst der Transformationsformeln leicht construiren lässt. Hierbei ist hervorzuheben, dass sowohl bei dieser Figur wie in der Folge die einzelnen Linien durch verschiedenerlei Punktirung gerade so unterschieden sind, wie schon die entsprechenden Linien im Polygon der s-Ebene.

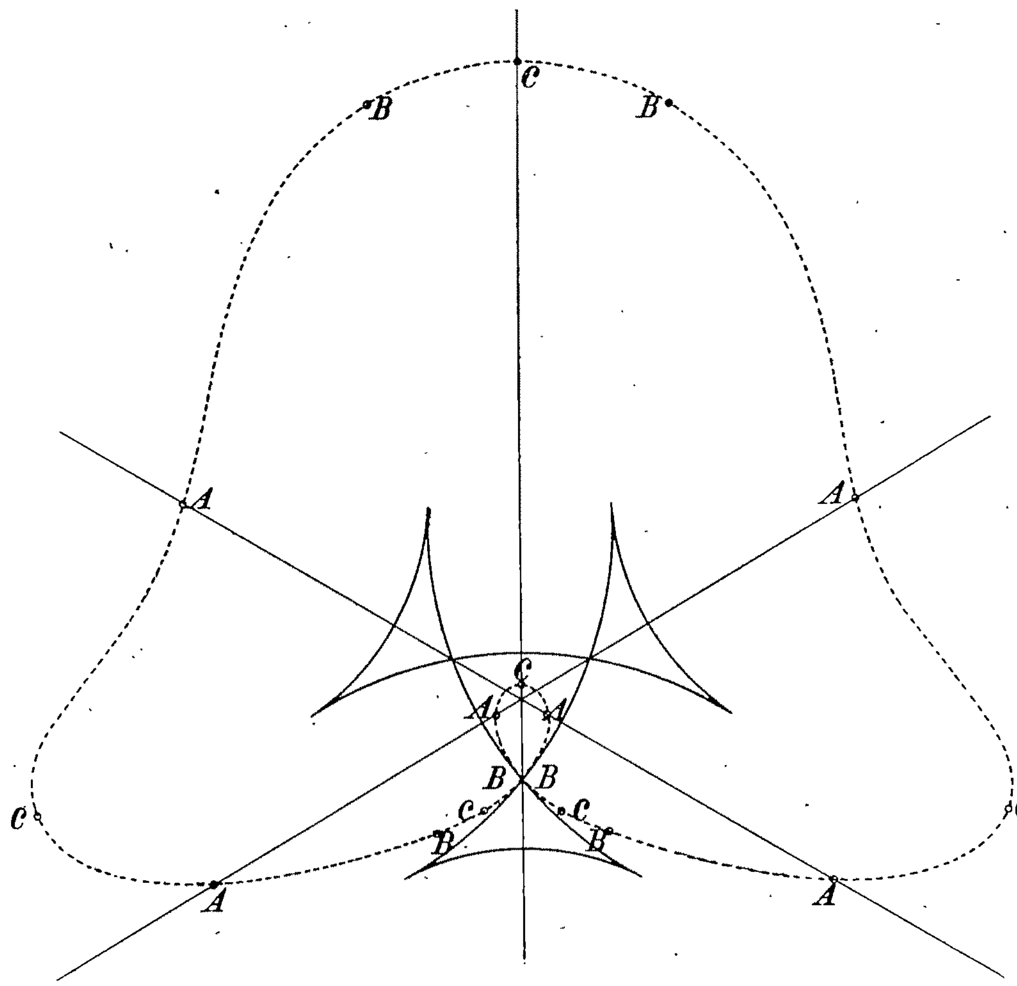


FIG. 9.

§17.

Die Symmetrielinien der dritten Art.

Die Symmetrielinien der dritten Art im Fundamentalpolygon hatten mit der Grundlinie keinen Punkt gemein. Die entsprechenden Curven der mehr-

fachen Ueberdeckung der Ebene dürfen also den reellen Curvenzug unserer Figur nicht treffen. Jedes Paar im Fundamentalpolygon symmetrisch liegender Curven dieser Art schneidet sich in zwei symmetrisch liegenden Punkten c. Wir schliessen daraus, dass die entsprechende Curve der mehrfachen Ueberdeckung jedenfalls einen reellen Doppelpunkt besitzt.

Als Repräsentanten dieser Art wählen wir diejenige Curve, welche zugleich die beiden symmetrisch liegenden Linien des Fundamentalpolygons, die aus der Grundlinie bei den Transformationen S^3TS, bezw. S^4TS^6 hervorgehen, vertritt. Die entsprechende Collineation S^3TS lautet:

$$\begin{aligned}\rho\lambda' &= \gamma\ (A\gamma^3\lambda + B\gamma^5\mu + C\gamma^6\nu),\\ \rho\mu' &= \gamma^4(B\gamma^3\lambda + C\gamma^5\mu + A\gamma^6\nu),\\ \rho\nu' &= \gamma^2(C\gamma^3\lambda + A\gamma^5\mu + B\gamma^6\nu)\end{aligned}$$

und die Collineation S^4TS^6 ist die conjugirt imaginäre Collineation. Die Combination dieser beiden Collineationen in der vorgeschriebenen Weise ergibt die folgende quadratische Transformation:

$$\begin{aligned}\sigma u &= -A(\lambda^2BC + \mu^2CA + \nu^2AB + \mu\nu B(A+1) + \nu\lambda B(C-1) + \lambda\mu C(A+1)),\\ \sigma v &= -C(\lambda^2CA + \mu^2AB + \nu^2BC + \mu\nu B(C+1) + \nu\lambda A(C+1) + \lambda\mu A(B-1)),\\ \sigma w &= -B(\lambda^2AB + \mu^2BC + \nu^2CA + \mu\nu C(A-1) + \nu\lambda A(B+1) + \lambda\mu C(B+1)).\end{aligned}$$

Wir haben sofort aus diesen Formeln:

$$\begin{aligned}-\sigma(u+v+w) &= BC\lambda^2 + AB\mu^2 + CA\nu^2 - BC\mu\nu - AB\nu\lambda - CA\lambda\mu\\ &= (BC\lambda + AB\mu + CA\nu)(\lambda + \mu + \nu).\end{aligned}$$

Haben nun die Kegelschnitte: $u = 0$, $v = 0$, $w = 0$ der λ, μ, ν-Ebene gemeinschaftliche Schnittpunkte, so muss der zerfallende Kegelschnitt

$$u + v + w = 0$$

auch durch dieselben Punkte laufen. Wir finden in der That, dass die Kegelschnitte drei reelle Schnittpunkte haben, und zwar liegt der eine:

$$\lambda : \mu : \nu = BC : AB : CA$$

auf der unendlich weiten Geraden

$$\lambda + \mu + \nu = 0;$$

die beiden anderen liegen auf der Perspectivitätsaxe

$$BC\lambda + AB\mu + CA\nu = 0,$$

und ihre Coordinaten sind bezw.

$$\lambda : \mu : \nu = A(B \pm \sqrt{C+1}) : CA : -B(A + B \pm \sqrt{C+1}).$$

Von diesen drei Fundamentalpunkten der Transformation liegt keiner auf der Grundcurve. Die transformirte Curve ist also, algebraisch zu reden, eine Curve achter Ordnung mit drei vierfachen Punkten. Um die Gleichung derselben zu erhalten, verfahren wir folgendermassen. Aus den Transformationsformeln folgt sofort:

$$-\sigma(uB - vA) = (BC\lambda + AB\mu + CA\nu)(B\nu - A\lambda)$$
$$-\sigma(vA - wC) = (BC\lambda + AB\mu + CA\nu)(A\lambda - C\mu).$$

Es ist also:

$$\frac{uB - vA}{vA - wC} = \frac{\nu B - \lambda A}{\lambda A - \mu C},$$

und wir können schreiben:

$$A\lambda = \tau . Av + x,$$
$$C\mu = \tau . Cw + x,$$
$$B\nu = \tau . Bu + x,$$

wo das Verhältnis $\tau : x$ noch zu bestimmen ist. Es besteht zunächst die Relation:

$$\lambda + \mu + \nu = \tau(u + v + w) + x\left(\frac{1}{A} + \frac{1}{C} + \frac{1}{B}\right)$$
$$= \tau(u + v + w)$$

und auch:

$$BC\lambda + AB\mu + CA\nu = \tau(BCv + ABw + CAu) + 2x.$$

Nun hatten wir schon die Gleichung:

$$-\sigma(u + v + w) = (BC\lambda + AB\mu + CA\nu)(\lambda + \mu + \nu).$$

Es ist also:

$$-\sigma(u + v + w) = \tau(u + v + w)[\tau(BCv + ABw + CAu) + 2x]$$

woraus folgt, da $(u + v + w)$ im Allgemeinen nicht gleich Null ist:

$$-\sigma = \tau^2(BCv + ABw + CAu) + 2\tau x.$$

Ferner ist, aus den Transformationsformeln:

$$-\sigma(BCu + ABv + CAw) = A^2B\mu\nu + C^2A\nu\lambda + B^2C\lambda\mu$$
$$= \tau^3(A^2Bwu + C^2Auv + B^3Cvw)$$
$$+ \tau x\left[\frac{A^2B + C^3}{C}u + \frac{C^2A + B^3}{B}v + \frac{B^2C + A^3}{A}w\right].$$

Bei Elimination von σ ergibt sich:

$$\frac{x}{\tau} = \frac{A^2Bwu + C^2Auv + B^2Cvw - (BCv + ABw + CAu)(BCu + ABv + CAw)}{-(CAu + BCv + ABw)}.$$

Setzt man also

$$\rho\tau = CAu + BCv + ABw,$$

so ist

$$\rho\varkappa = ABC(Cu + Bv + Aw) + B^2C^2vw + A^2B^2wu + C^2A^2uv,$$

und wir bekommen die endgültigen Formeln der Umkehrtransformation in der Form:

$$\rho\lambda = \rho\left(\tau v + \frac{\varkappa}{A}\right)$$
$$= C(BCu^2 + CAv^2 + ABw^2 + A(B-1)vw + B(C+1)wu + A(C+1)uv)$$
$$\rho\mu = \rho\left(\tau w + \frac{\varkappa}{C}\right)$$
$$= B(CAu^2 + ABv^2 + BCw^2 + C(B+1)vw + C(A-1)wu + A(B+1)uv)$$
$$\rho\nu = \rho\left(\tau u + \frac{\varkappa}{B}\right)$$
$$= A(ABu + BCv + CAw + C(A+1)vw + B(A+1)wu + B(C-1)uv).$$

Die Gleichung der transformirten Curve geht bei Einsetzung dieser Werte für $\lambda:\mu:\nu$ aus der Gleichung der Grundcurve hervor. Die Form der Curve ist in Figur 10 angegeben, wobei man ohne Weiteres sieht, dass die beiden mehrfachen Punkte der Curve nicht Doppelpunkte der einzelnen Symmetrielinien auf der Fläche sind, denn die beiden Aeste der einzelnen Curve laufen jedesmal auf verschiedenen Blättern durch die Doppelpunkte hindurch.

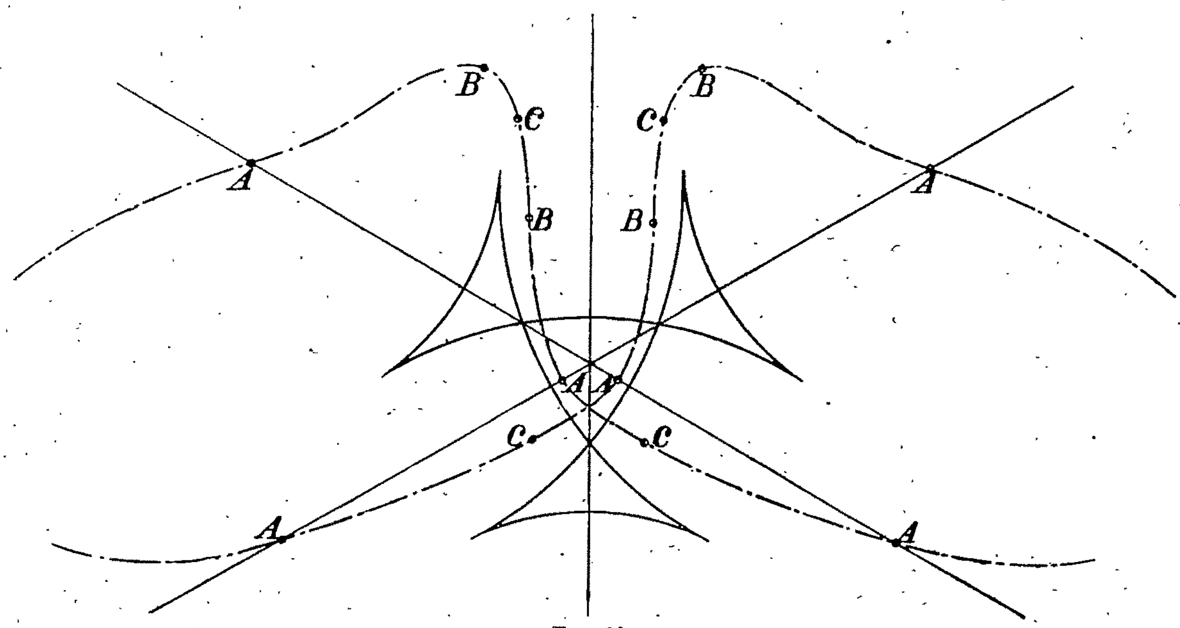


FIG. 10.

§18.

Die Symmetrielinien der vierten Art.

Als Repräsentanten dieser Art wählen wir diejenige Curve, die die beiden Linien des Fundamentalpolygons, welche aus der Grundlinie bei den Substitutionen S, bezw. S^6 hervorgehen, vertritt. Die entsprechenden Collineationen S und S^6 lauten:

$$S: \rho\lambda' = \gamma\lambda, \quad \rho\mu' = \gamma^4\mu, \quad \rho\nu' = \gamma^2\nu,$$
$$S^6: \rho\lambda'' = \gamma^6\lambda, \quad \rho\mu'' = \gamma^3\mu, \quad \rho\nu'' = \gamma^5\nu.$$

Die entsprechende quadratische Transformation für den reellen Curvenzug heisst dann einfach:

$$\sigma u = A\mu\nu,$$
$$\sigma v = C\nu\lambda,$$
$$\sigma w = B\lambda\mu.$$

Die drei Fundamentalpunkte der Transformation sind einfach die Ecken des Coordinatendreiecks, und die Transformation lässt sich bekanntlich sofort in die folgende Gestalt umkehren:

$$\tau\lambda = Avw,$$
$$\tau\mu = Cwu,$$
$$\tau\nu = Buv.$$

Substituiren wir diese Werte von λ, μ, ν in der Gleichung

$$\lambda^3\mu + \mu^3\nu + \nu^3\lambda = 0,$$

so erhalten wir

$$uvw\,(A^3C\,v^2w^3 + C^3B\,w^2u^3 + B^3A\,u^2v^3) = 0.$$

Hierbei ist der Factor uvw unwesentlich, denn es entsprechen die drei Geraden $u = 0$, $v = 0$, $w = 0$ bezw. den drei Fundamentalpunkten der Transformation. Die gesuchte Gleichung der Symmetrielinie lautet demnach:

$$A^3C\,v^2w^3 + C^3B\,w^2u^3 + B^3A\,u^2v^3 = 0.$$

Diese Gleichung stellt eine Curve fünfter Ordnung mit drei reellen Spitzen dar. Dieselben sind, wie sofort aus der Form der Gleichung zu ersehen, eben die drei Ecken des Coordinatendreiecks, und die Spitzentangenten sind die Seiten desselben.

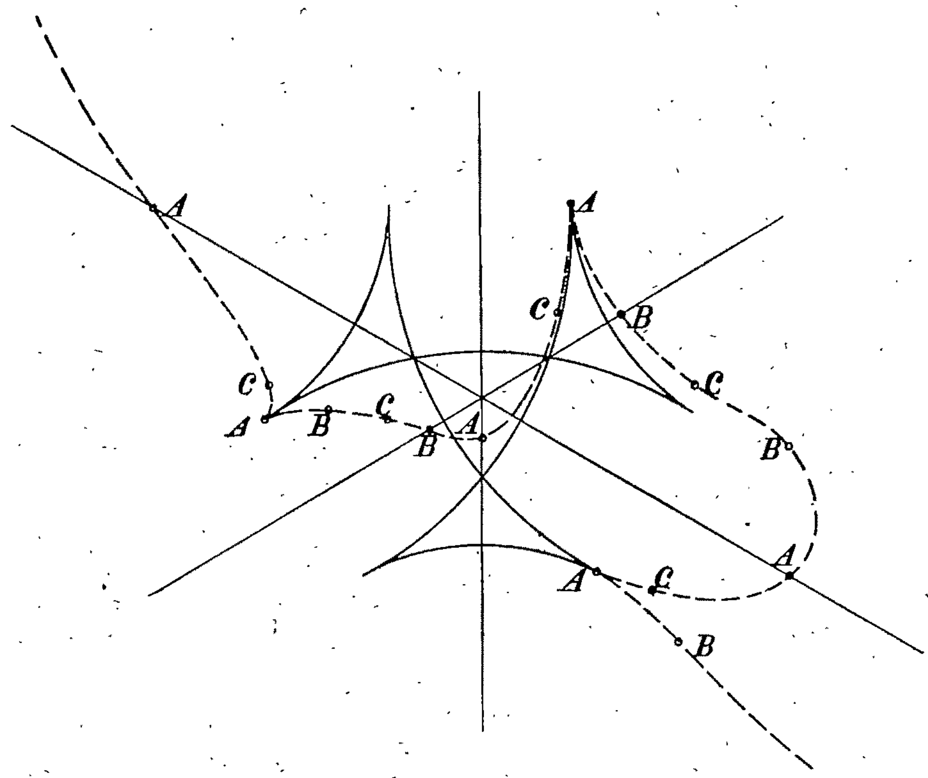


FIG. 11.

Man erhält die übrigen Symmetrielinien dieser Art, indem man diese Curve um den Mittelpunkt der Figur um 120°, resp. 240° dreht, oder gegen die drei Symmetrieaxen spiegelt. Die Gleichungen derselben ergeben sich aus der gefundenen Gleichung bei einfacher cyclischer Vertauschung der Buchstaben, resp. bei Ersetzung der Variabeln u, v, w beispielsweise durch $Au + Bv + Cw$, bezw. $Bu + Cv + Aw$ und $Cu + Av + Bw$.

FÜNFTER ABSCHNITT.

Erbringung der noch ausstehenden Beweise und Schilderung der endgültigen Figur.

Es handelt sich jetzt in der Hauptsache um den Beweis solcher Sätze, die in allgemeiner Form im oben citirten Aufsatze von Herrn Klein* bereits aufgestellt worden sind, und die wir in ihrer Bedeutung für unsere Curve schon in den vorangehenden Abschnitten erläutert haben. Der Beweis derselben wird

* Math. Annalen, Bd. 10, l. c.

hier einfacher, weil wir uns immer auf die Beziehung zum Fundamentalpolygon stützen können. Es soll übrigens im letzten Paragraphen ein genauer Nachweis dieser Beziehung sich vorfinden.

§19.

Zusammenhang der Ueberdeckung längs des reellen Curvenzuges.

Schon in §12 ist der Satz ausgesprochen worden: *Langs des reellen Curvenzuges hängen immer zwei Blätter zusammen.* Dieser Satz lässt sich aus einfacher Betrachtung der Lage zweier im Fundamentalpolygon in Bezug auf die Grundlinie symmetrisch liegender Punkte herstellen. Denn betrachten wir irgend zwei Dreiecke, welche einen Teil der Grundlinie zur gemeinschaftlichen Begrenzung haben, so lauten symmetrisch liegende Punkte

$$\tau = \pm a + bi,$$

und die entsprechenden Werte von $\lambda : \mu : \nu$ sind conjugirt imaginär und werden von einem und demselben Punkt der uvw-Ebene dargestellt. Die beiden Dreiecke werden also durch zwei übereinanderliegende Dreiecke der mehrfachen Ueberdeckung dargestellt, und da die beiden Dreiecke des Fundamentalpolygons langs der Grundlinie zusammenhängen, müssen sie auf der mehrfachen Ueberdeckung längs des reellen Curvenzuges zusammenhängen.

Welche zwei Blätter aber längs der verschiedenen Teile des reellen Curvenzuges zusammenhängen, ist auch ohne weiteres klar. Denn die mehrfache Ueberdeckung der Ebene entspricht dem Fundamentalpolygon Punkt für Punkt, und das zusammengeheftete Polygon besitzt keinen Rand. Die mehrfache Ueberdeckung der Ebene kann also auch nicht berandet sein, und *es müssen also jedesmal diejenigen zwei Blätter, welche an einem bestimmten Stücke des reellen Curvenzuges aufzuhören scheinen, dort zusammenhängen.*

§20.

Der Doppelverzweigungspunkt.

Damit nun die ganze Figur ein zusammenhängendes Ganze bilde, müssen die grossen Blätter irgendwo in die kleinen übergehen, und es muss also innerhalb des inneren Teils der Figur ein Verzweigungspunkt existiren. Während nun im Allgemeinen von jedem Punkte dieses Teils der Ebene vier getrennte imaginäre Tangenten an die Curve laufen, so laufen jedoch von einem Punkte,

nämlich vom isolirten Doppelpunkte aus nur zwei solche. Also müssen die Blätter in diesem Punkte paarweise zusammenhängen, und zwar besteht jedes solche Paar aus einem grossen und einem kleinen Blatt. Es wird hiernach behauptet und sogleich noch näher bewiesen: *der isolirte Doppelpunkt ist ein Verzweigungspunkt der mehrfachen Ueberdeckung der Ebene.*

Um diesen Satz zu beweisen, untersuchen wir das Verhalten der durch den betreffenden Punkt hindurchlaufenden Symmetrielinien der ersten Art. Wir haben schon gezeigt, dass dieselben bezw. auf den drei Symmetrieaxen der Figur liegen. Laufen sie einfach durch diesen Punkt hindurch, so ist derselbe kein Verzweigungspunkt. Ich werde aber zeigen, dass sie nicht durch den Punkt hindurchlaufen, sondern bloss bis an den Punkt heran, und dann gleich wieder zurück. Die beiden Teile einer Symmetrielinie dürfen aber auf der mehrfachen Ueberdeckung der Ebene nicht zusammenfallen; sie müssen also in verschiedenen Blättern liegen, und zwar liegt der eine Teil in einem grossen Blatt, der andere Teil in dem zugehörigen kleinen Blatt. Wenn nun keine Verzweigung statt fände, so wäre es möglich, auf einem dem Doppelpunkt unendlich nahe liegenden Wege von der einen Seite der Symmetrielinie ohne Ueberschreitung derselben an die andere Seite zu gelangen. Dies ist aber aus der Natur der Sache unmöglich, und der Doppelpunkt ist also ein Verzweigungspunkt.

Dass die Symmetrielinien nicht durch den Doppelpunkt hindurchlaufen, wird am einfachsten folgendermassen bewiesen. Wir wählen beispielsweise die Symmetrielinie, die auf der Symmetrieaxe

$$BCu + ABv + CAw = 0$$

liegt. Dieselbe geht aus der Grundcurve hervor durch folgende Transformation (§15):

$$\begin{aligned}
\sigma u &= A(\lambda^2 BC + \mu^2 CA + \nu^2 AB + \mu\nu C^2 - \nu\lambda(B^2 + C^2) + \lambda\mu B^2),\\
\sigma v &= C(\lambda^2 CA + \mu^2 AB + \nu^2 BC + \mu\nu A^2 + \nu\lambda B^2 - \lambda\mu(A^2 + B^2)),\\
\sigma w &= B(\lambda^2 AB + \mu^2 BC + \nu^2 CA - \mu\nu(C^2 + A^2) + \nu\lambda C^2 + \lambda\mu A^2),
\end{aligned}$$

und zwar gehen die beiden Berührungspunkte die auf der Doppeltangente

$$A^2\lambda + C^2\mu + B^2\nu = 0$$

liegen, in den Doppelpunkt $1:1:1$ über. Von diesen beiden Berührungspunkten ist der eine

$$\lambda:\mu:\nu = \frac{1}{C(\gamma^5\alpha + \gamma^2\alpha^2)} : \frac{1}{B(\gamma^6\alpha + \gamma\alpha^2)} : \frac{1}{A(\gamma^3\alpha + \gamma^4\alpha^2)}.$$

Entwickeln wir nun $u:v$ in der Nähe dieses Punktes. Es ist

$$\frac{u'}{v'} = \frac{u}{v} + d\left(\frac{u}{v}\right) = \frac{u}{v} + \frac{vdu - udv}{v^2} = 1 + \frac{du - dv}{v},$$

wenn $u:v = 1$. Est ist nun:

$$\begin{aligned}\sigma . du = A\{ & d\lambda(2BC\lambda + B^2\mu - (B^2 + C^2)\nu) \\ & + d\mu(B^2\lambda + 2CA\mu + C^2\nu) \\ & + d\nu(-(B^2 + C^2)\lambda + C^2\mu + 2AB\nu)\} \\ & + \ldots\ldots, \\ \sigma . dv = C\{ & d\lambda(2CA\lambda - (A^2 + B^2)\mu + B^2\nu) \\ & + d\mu(-(A^2 + B^2)\lambda + 2AB\mu + A^2\nu) \\ & + d\nu(B^2\lambda + A^2\mu + 2BC\nu)\} \\ & + \ldots.\end{aligned}$$

woraus folgt:

$$\begin{aligned}\sigma(du - dv) = -\{ & d\lambda(2A^2\lambda + C^2(B+1)\mu + A^2\nu) \\ & + d\mu(C^2(B+1)\lambda + 2C^2(B+1)\mu + B^2C\nu) \\ & + d\nu(A^2\lambda + B^2C\mu + 2B^2C\nu)\} \\ & + \text{Glieder höherer Ordnung.}\end{aligned}$$

Eine einfache Rechnung ergibt, dass diese Coefficienten mit $A^2 \colon C^2 \colon B^2$ proportional sind. Für Punkte der Curve in unmittelbarer Nähe des betreffenden Berührungspunktes ist aber:

$$A^2 d\lambda + C^2 d\mu + B^2 d\nu = 0,$$

denn diese Gleichung ergibt die Richtung der Tangente in diesem Punkt. Die Reihenentwickelung für $u:v$ fängt also für Punkte der Symmetrielinie mit Gliedern der zweiten Ordnung an, und $d\,\frac{u}{v}$ hat denselben Wert, ob wir in der einen oder in der anderen Richtung auf der Grundcurve fortfahren. Dasselbe gilt für $u:w$, wie leicht aus einer ähnlichen Berechnung zu ersehen, und damit ist unser Satz bewiesen.

Wir sind nunmehr im Stande zu entscheiden, welche Teile einer Symmetrieaxe eigentlich als Symmetrielinie erster Art auftreten, und wie überhaupt sich die letztere über die verschiedenen Blätter der mehrfachen Ueberdeckung hinüberzieht.

Das Verhalten wird offenbar durch folgende Zeichnung dargestellt, wobei ich mir die kleinen Blätter zwischen den grossen Blättern gelegen denke, und die

verschiedenen Teile der Symmetrielinie, welche eigentlich genau über einander liegen, der Deutlichkeit halber etwas getrennt gezeichnet habe.

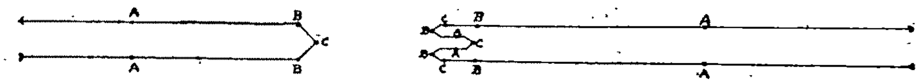


FIG. 12.

§21.

Winkelabbildung auf der Figur.

Es bleibt jetzt noch übrig, bei der Abbildung des Fundamentalpolygons auf die mehrfache Ueberdeckung der Ebene das Gesetz zu ermitteln, welches an Stelle der gewöhnlichen Regel tritt, derzufolge die Beziehung zweier Riemann'scher Flächen conform sein soll. Es ist nämlich jede Abbildung einer Fläche auf eine zweite immer winkeltreu, sofern man nicht nur die Richtungen, die den Winkel einschliessen, sondern auch die Richtungen, in Bezug auf die derselbe im projectiven Sinne definirt wird, mit transformirt denkt. Denn, sind p, q irgend krummlinige Coordinaten in der Ebene, und sind zwei imaginäre Fortschreitungsrichtungen gegeben durch

$$dS = Edp^2 + 2Fdp\,dq + Gdq^2 = 0,$$

so nennt man den "auf diese Richtungen bezogenen" Winkel zweier Bogenelemente d', d'':

$$\phi = \text{arc cos}\,\frac{Ed'p\,d''p + F(d'p\,d''q + d''p\,d'q) + Gd'q\,d''q}{\sqrt{Ed'p^2 + 2Fd'pd'q + Gd'q^2}\sqrt{Ed''p^2 + 2Fd''pd''q + Gd''q^2}}$$

und dieser Wert von ϕ bleibt, wie sofort ersichtlich, bei *allen analytischen Abbildungen* erhalten.

Der wirkliche Winkel wird aus dieser Formel entstehen, wenn die angenommenen Richtungen die Richtungen der nach den imaginären Kreispunkten laufenden Geraden sind, also, wenn p, q insbesondere *orthogonale* Coordinaten sind, durch

$$dp^2 + dq^2 = 0$$

gegeben sind, also in der τ-Ebene durch

$$dx^2 + dy^2 = 0,$$

wenn $\tau = x + iy$ gesetzt wird. Es ist nun

$$dx^2 + dy^2 = -(dx + idy)(-dx + idy),$$

und es entsteht die Frage, welche (imaginären) Richtungen auf der mehrfachen Ueberdeckung der uvw-Ebene diesen (imaginären) Richtungen in der τ-Ebene entsprechen. Jeder Punkt der mehrfachen Ueberdeckung ist bestimmt als der Schnittpunkt zweier conjugirt imaginärer Tangenten, deren Coordinaten $\lambda : \mu : \nu$ und $\lambda' : \mu' : \nu'$ von $\tau = x + iy$, bezw. von $\tau' = -x + iy$ abhängen. Ist also $d\tau = 0$, so liegt die entsprechende Fortschreitungsrichtung auf der einen dieser Tangenten; ist dagegen $d\tau' = 0$, so liegt dieselbe auf der anderen. Dem Winkel der τ-Ebene bezogen auf die Richtungen

$$dx^2 + dy^2 = 0$$

entspricht daher der Winkel bezogen auf die zwei imaginären Tangenten, die vom betreffenden Punkte aus an die Curve vierter Classe hinlaufen, und das ist das zu beweisende Resultat.

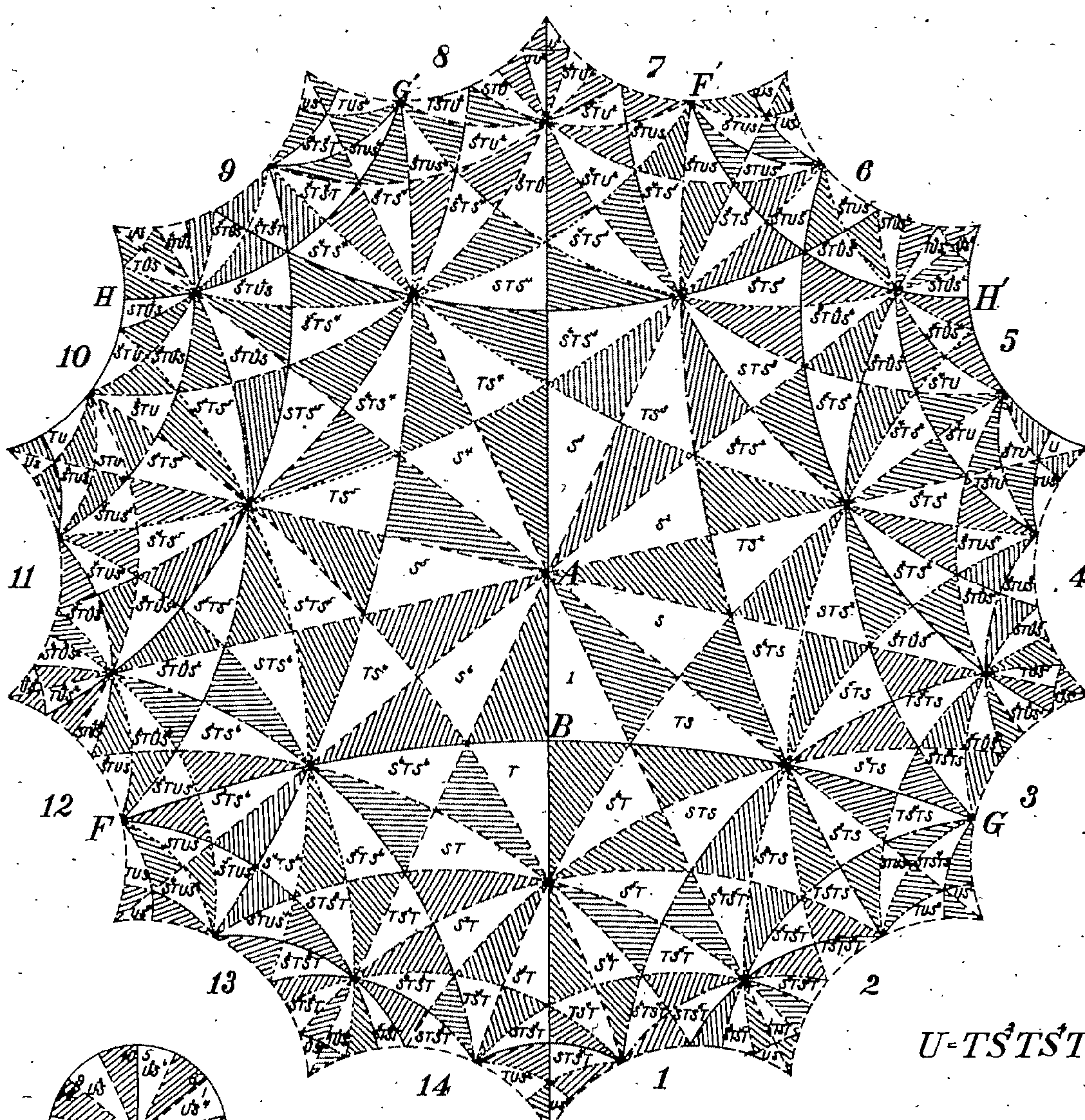
§22.

Definitive Gestalt der Figur.

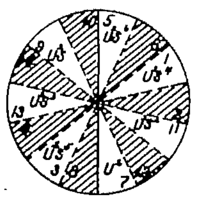
Wir sind nunmehr zur endgültigen Figur der mehrfachen Ueberdeckung der Ebene geführt, welche in Tafel II gegeben ist. Hierbei sind nur die oberen Blätter gezeichnet, da die unteren Blätter diesen genau ähnlich sind. Ich habe auch der Deutlichkeit halber die beiden oberen Blätter getrennt gezeichnet. Dieselben sind aber nach den früheren Angaben längs der (stark gezeichneten) Verzweigungsschnitte verbunden zu denken. Es bleibt uns übrig, die genaue Beziehung derselben zum Fundamentalpolygon nachzuweisen. Wir haben zunächst schon gezeigt, dass sich jede Symmetrielinie hier wieder vorfindet, und dass ferner die merkwürdigen Punkte in der bekannten Reihenfolge auf denselben liegen.

Hier wieder ist die ganze Oberfläche durch die Symmetrielinien in 2.168 Dreiecke geteilt, welche, dem Fundamentalpolygon entsprechend, alternirend schraffirt und nicht schraffirt sind. Um die Beziehung ganz genau wiederzugeben, habe ich erstens die einzelnen Classen von Symmetrielinien durch Punktirung so unterschieden wie die entsprechenden Symmetrielinien des Fundamentalpolygons; dann habe ich auch die einzelnen Dreiecke mit denselben Symbolen bezeichnet, mit denen die entsprechenden Dreiecke des Fundamentalpolygons bezeichnet sind, wobei jedesmal das Symbol eines schraffirten Dreiecks bezüglich das darunterliegende nicht schraffirte Dreieck des unteren Blattes zu verstehen ist.

INHALTSVERZEICHNIS.



$U = TS^3TS^4TS^4$

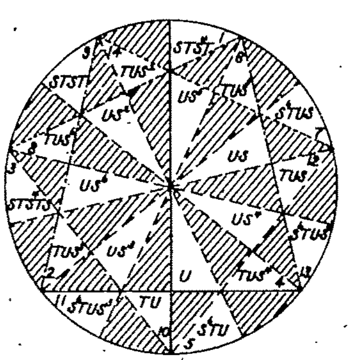


Ecken der einen Art

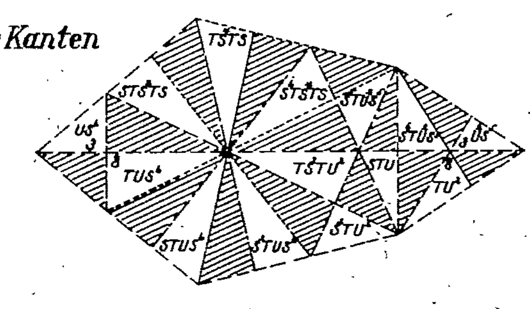
Das S-Polygon

Zusammengehörigkeit der Kanten

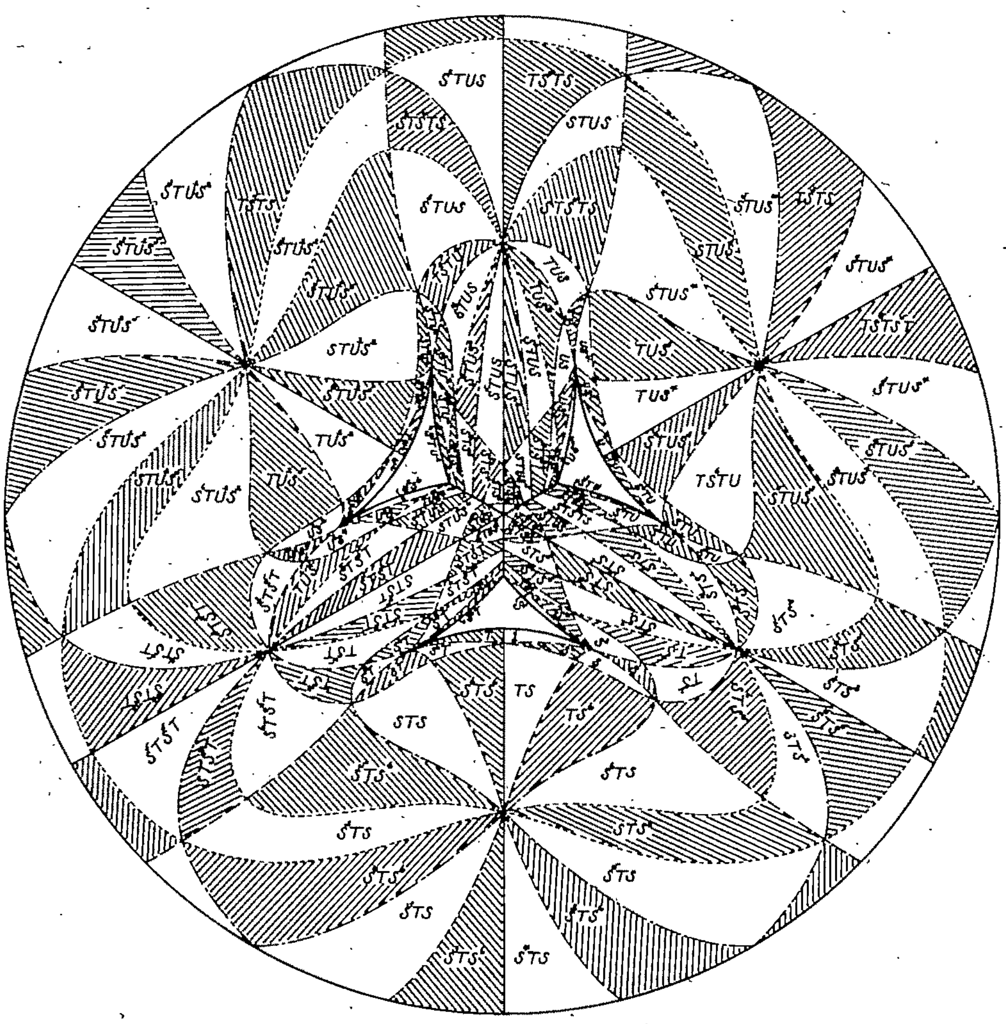
1	an	6
3	„	8
5	„	10
7	„	12
9	„	14
11	„	2
13	„	4

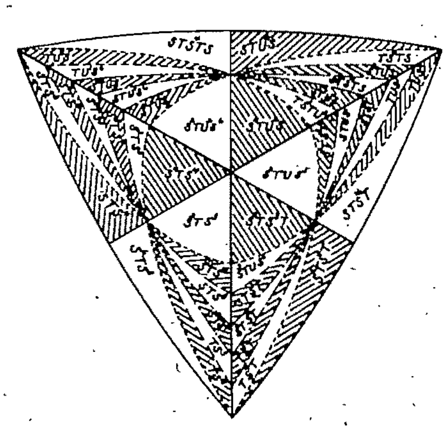


Ecken der anderen Art.



Zusammenhang der Kanten 3 und 8





Das kleine Blatt.

On a Soluble Quintic Equation.

By Prof. Cayley.

Mr. Young, in his paper, "Soluble Quintic Equations with Commensurable Coefficients," *Am. Math. Jour.* X (1888), pp. 99–130, has given, in illustration of his general theory of the solution of soluble quintic equations (founded upon a short note by Abel) no less than twenty instances of the solution of a quintic equation with purely numerical coefficients, having a solution of the form $\sqrt[5]{A}+\sqrt[5]{B}+\sqrt[5]{C}+\sqrt[5]{D}$, where A, B, C, D are numerical expressions involving only square roots. But the solutions are not presented in their most simple form: thus in example 1, $x^5+3x^3+2x-1=0$, the expression involves a radical

$$\sqrt{\frac{47}{8}(21125+9439\sqrt{5})}\text{: here } (21125+9439)\sqrt{5},\ =\sqrt{5}\,(9439+4225\sqrt{5}),$$
$$=\sqrt{5}\,.\,\frac{1}{2}(18+5\sqrt{5})^2(1+\sqrt{5})^2(2+\sqrt{5}),$$

so that, taking out the roots of the squared factors, we have as the proper form of the radical the very much more simple form $\sqrt{47(2+\sqrt{5})\sqrt{5}}$; where observe that $(2+\sqrt{5})(2-\sqrt{5})=-1$, and thence $(2+\sqrt{5})\sqrt{-47(2-\sqrt{5})\sqrt{5}}$ $=\sqrt{47(2+\sqrt{5})\sqrt{5}}$, viz. the conjugate radicals $\sqrt{-47(2-\sqrt{5})\sqrt{5}}$ and $\sqrt{47(2+\sqrt{5})\sqrt{5}}$ differ only by a factor $2+\sqrt{5}$ which is rational in 1 and $\sqrt{5}$. To avoid fractions I consider the foregoing equation under the form

$$x^5+3000x^3+20000x-100000=0,$$

and I will presently give the solution; but first I consider the general theory. Writing

$$\begin{array}{lll} A=\alpha^5, & A'=\alpha^2\gamma\,, & A''=\alpha^3\beta, \\ B=\beta^5, & B'=\alpha\beta^2, & B''=\beta^3\delta\,, \\ C=\gamma^5, & C'=\gamma^2\delta\,, & C''=\alpha\gamma^3, \\ D=\delta^5, & D'=\beta\delta^2, & D''=\gamma\delta^3\,, \end{array}$$

we have $A'D' = \alpha^2\delta^2\beta\gamma$, $B'C' = \alpha\delta\beta^2\gamma^2$. Also

$$A'' = \frac{A'B'}{\beta\gamma}, \quad B'' = \frac{B'D'}{\alpha\delta}, \quad C'' = \frac{A'C'}{\alpha\delta}, \quad D'' = \frac{C'D'}{\beta\gamma},$$

which determine A'', B'', C'', D'' in terms of A', B', C', D', $\alpha\delta$, $\beta\gamma$; and then

$$A = \frac{A'A''}{\beta\gamma}, \quad B = \frac{B'B''}{\alpha\delta}, \quad C = \frac{C'C''}{\alpha\delta}, \quad D = \frac{D'D''}{\beta\gamma},$$

which give A, B, C, D.

If now we assume $x = \alpha + \beta + \gamma + \delta$, and regard $A, B, C, D, A', B', C', D', A'', B'', C'', D''$, $\alpha\delta$, $\beta\gamma$ each as a rational function, we may express x, x^2, x^3, x^5 each of them by means of rational functions or of rational functions multiplied into α, β, γ, δ respectively: thus,

$$\begin{aligned}
x &= \alpha + \beta + \gamma + \delta && = \alpha + \beta + \gamma + \delta, \\
x^2 &= \alpha^2 + \beta^2 + \gamma^2 + \delta^2 && = \frac{A'\beta}{\beta\gamma} + \frac{B'\delta}{\alpha\delta} + \frac{C'\alpha}{\alpha\delta} + \frac{D'\gamma}{\beta\gamma}; \\
&+ 2\alpha\beta + 2\alpha\gamma + 2\alpha\delta + 2\beta\gamma + 2\beta\delta + 2\gamma\delta && + \frac{2B'\gamma}{\beta\gamma} + \frac{2A'\delta}{\alpha\delta} + 2\alpha\delta + 2\beta\gamma + \frac{2D'\alpha}{\alpha\delta} + \frac{2C'\beta}{\beta\gamma},
\end{aligned}$$

etc., and we thus obtain

$$\begin{aligned}
& x^5 + qx^3 + rx^2 + sx + t \\
&= A + B + C + D \\
&\quad + (20\alpha\delta + 30\beta\gamma)(A' + D') + 30(\alpha\delta + 20\beta\gamma)(B' + C') \\
&\qquad\qquad + q.3(A' + B' + C' + D') + r.2(\alpha\delta + \beta\gamma) + t \\
&+ \alpha\left\{5A'' + 5\frac{C'^2}{\alpha\delta} + \frac{5B''\beta\gamma}{\alpha\delta} + 10C'' + \frac{10D'^2}{\alpha\delta}\right. \\
&\qquad + 10\alpha^2\delta^2 + 20B'' + 20D'' + 30\beta^2\gamma^2 + 30D''\frac{\beta\gamma}{\alpha\delta} + 60\alpha\beta\gamma\delta \\
&\qquad \left. + q\left(\frac{B''}{\alpha\delta} + \frac{3D''}{\alpha\delta} + 3\alpha\delta + 6\beta\gamma\right) + r\left(\frac{C'}{\alpha\delta} + \frac{2D'}{\alpha\delta}\right) + s\right\} \\
&+ \beta\left\{5B'' + \frac{5A'^2}{\beta\gamma} + 5\frac{D''\alpha\delta}{\beta\gamma} + 10A'' + \frac{10C'^2}{\beta\gamma}\right. \\
&\qquad + 10\beta^2\gamma^2 + 20D'' + 20C'' + 30\alpha^2\delta^2 + 30C''\frac{\alpha\delta}{\beta\gamma} + 60\alpha\beta\gamma\delta \\
&\qquad \left. + q\left(\frac{D''}{\beta\gamma} + \frac{3C''}{\beta\gamma} + 3\beta\gamma + 6\alpha\delta\right) + r\left(\frac{A'}{\beta\gamma} + \frac{2C'}{\beta\gamma}\right) + s\right\}
\end{aligned}$$

$$+\gamma\left\{5C'' + \frac{5D'^2}{\beta\gamma} + 5\frac{A''\alpha\delta}{\beta\gamma} + 10D'' + \frac{10B'^2}{\beta\gamma}\right.$$
$$+ 10\beta^2\gamma^2 + 20A'' + 20B'' + 30\alpha^2\delta^2 + 30B''\frac{\alpha\delta}{\beta\gamma} + 60\alpha\beta\gamma\delta$$
$$\left. + q\left(\frac{A''}{\beta\gamma} + \frac{3B''}{\beta\gamma} + 3\beta\gamma + 6\alpha\delta\right) + r\left(\frac{D'}{\beta\gamma} + \frac{2B'}{\beta\gamma}\right) + s\right\}$$
$$+\delta\left\{5D'' + \frac{5B'^2}{\alpha\delta} + 5\frac{C''\beta\gamma}{\alpha\delta} + 10B'' + \frac{10A'^2}{\alpha\delta}\right.$$
$$+ 10\alpha^2\delta^2 + 20C'' + 20A'' + 30\beta^2\gamma^2 + 30A''\frac{\beta\gamma}{\alpha\delta} + 60\alpha\beta\gamma\delta$$
$$\left. + q\left(\frac{C''}{\alpha\delta} + \frac{3A''}{\alpha\delta} + 3\alpha\delta + 6\beta\gamma\right) + r\left(\frac{B'}{\alpha\delta} + \frac{2A'}{\alpha\delta}\right) + s\right\}.$$

If, then, $x^5 + qx^3 + rx^2 + sx + t = 0$, we have the rational term $= 0$, and the coefficients of $\alpha, \beta, \gamma, \delta$ each $= 0$; in the class of equations under consideration these last equations differ only in the signs of the radicals contained therein, so that one of them being satisfied identically, the others will be also satisfied. In particular, if $q = 0$, then $\alpha\delta + \beta\gamma = 0$: the rational term gives

$$A + B + C + D - 10\alpha\delta\,(A' + D' - B' - C') + t = 0,$$

and the term in α gives

$$5A'' + 15B'' + 10C'' - 10D'' + \frac{5}{\alpha\delta}\,(C'^2 + 2D'^2) + \frac{r}{\alpha\delta}\,(C' + 2D') - 20\alpha^2\delta^2 + s = 0.$$

For the equation $x^5 + 3000x^2 + 20000x - 100000 = 0$, the expression for the root is $x = \sqrt[5]{A} + \sqrt[5]{B} + \sqrt[5]{C} + \sqrt[5]{D}$, where

$$\begin{aligned}
A &= 39000 + 18200\sqrt{5} + (\ \ 1720 + 920\sqrt{5})\sqrt{235 + 94\sqrt{5}},\\
D &= 39000 + 18200\sqrt{5} + (-1720 - 920\sqrt{5})\sqrt{235 + 94\sqrt{5}},\\
B &= 39000 - 18200\sqrt{5} + (-1720 + 920\sqrt{5})\sqrt{235 - 94\sqrt{5}},\\
C &= 39000 - 18200\sqrt{5} + (\ \ 1720 - 920\sqrt{5})\sqrt{235 - 94\sqrt{5}},
\end{aligned}$$

and where also

$$\begin{aligned}
A' &= -150 - 70\sqrt{5} + (-10 - 2\sqrt{5})\sqrt{235 + 94\sqrt{5}},\\
D' &= -150 - 70\sqrt{5} + (\ \ 10 + 2\sqrt{5})\sqrt{235 + 94\sqrt{5}},\\
B' &= -150 + 70\sqrt{5} + (\ \ 10 - 2\sqrt{5})\sqrt{235 - 94\sqrt{5}},\\
C' &= -150 + 70\sqrt{5} + (-10 + 2\sqrt{5})\sqrt{235 - 94\sqrt{5}},
\end{aligned}$$

and

$$
\begin{aligned}
A'' &= -940 - 100\sqrt{5} + (-100 + 20\sqrt{5})\sqrt{235 + 94\sqrt{5}},\\
D'' &= -940 - 100\sqrt{5} + (100 - 20\sqrt{5})\sqrt{235 + 94\sqrt{5}},\\
B'' &= -940 + 100\sqrt{5} + (100 + 20\sqrt{5})\sqrt{235 - 94\sqrt{5}},\\
C'' &= -940 + 100\sqrt{5} + (-100 - 20\sqrt{5})\sqrt{235 - 94\sqrt{5}}.
\end{aligned}
$$

The foregoing forms are in some respects the most convenient, but it is to be observed that we have

$$
\begin{aligned}
A &= 2600\sqrt{5}(1+\sqrt{5})(2+\sqrt{5})+40(1+\sqrt{5})(18+5\sqrt{5})\sqrt{47\sqrt{5}(2+\sqrt{5})}, \text{ etc.,}\\
A' &= -10\sqrt{5}(1+\sqrt{5})(2+\sqrt{5}) - 2\sqrt{5}(1+\sqrt{5})\sqrt{47\sqrt{5}(2+\sqrt{5})}, \text{ etc.,}\\
A'' &= 20(1-\sqrt{5})(18+13\sqrt{5}) + 20\sqrt{5}(1-\sqrt{5})\sqrt{47\sqrt{5}(2\sqrt{+5})}, \text{ etc.,}
\end{aligned}
$$

or putting for shortness

$$\sqrt{Q} = \sqrt{47\sqrt{5}(2+\sqrt{5})},\quad \sqrt{Q_1} = \sqrt{-47\sqrt{5}(2-\sqrt{5})},$$

(so that, according to a foregoing remark, we have $(2+\sqrt{5})\sqrt{Q} = \sqrt{Q_1}$), then we have

$$
\begin{aligned}
A &= 40(1+\sqrt{5})\{65\sqrt{5}(2+\sqrt{5}) + (18+5\sqrt{5})\sqrt{Q}\}, \text{ etc.,}\\
A' &= -2\sqrt{5}(1+\sqrt{5})\{5(2+\sqrt{5}) + \sqrt{Q}\}, \text{ etc.,}\\
A'' &= 20(1-\sqrt{5})\{18 + 13\sqrt{5} + \sqrt{5}\sqrt{Q}\}, \text{ etc.,}
\end{aligned}
$$

where observe that the term $2+\sqrt{5}$ is a factor of Q.

Starting from the values of A', B', C', D', we have

$$
\begin{aligned}
A' &= -2\sqrt{5}(1+\sqrt{5})\{5(2+\sqrt{5}) + \sqrt{Q}\},\\
D' &= -2\sqrt{5}(1+\sqrt{5})\{5(2+\sqrt{5}) - \sqrt{Q}\},
\end{aligned}
$$

and therefore

$$A'D' = 20(1+\sqrt{5})^2(2+\sqrt{5})\{25(2+\sqrt{5}) - 47\sqrt{5}\},$$

where the last factor is

$$= 50 - 22\sqrt{5},\ = -2\sqrt{5}(11 - 5\sqrt{5}),\ = -\sqrt{5}(1-\sqrt{5})^2(2-\sqrt{5}).$$

Hence

$$A'D' = -\sqrt{5}.20(-4)^2(-1) = 320\sqrt{5},$$

that is,

$$A'D' = (\alpha\delta)^2\beta\gamma = 320\sqrt{5}, \text{ and similarly } B'C' = \alpha\delta(\beta\gamma)^2 = -320\sqrt{5},$$

whence

$$\alpha\delta = -4\sqrt{5},\quad \beta\gamma = 4\sqrt{5}. \qquad (\alpha\delta + \beta\gamma = 0, \text{ as above.})$$

We have, moreover,

$$A' = -2\sqrt{5}(1+\sqrt{5})\{5(2+\sqrt{5})+\sqrt{Q}\},$$
$$B' = 2\sqrt{5}(1-\sqrt{5})\{5(2-\sqrt{5})-\sqrt{Q_1}\},$$

and thence

$$A'B' = 80\{-25-\sqrt{QQ_1}+5(2-\sqrt{5})\sqrt{Q}-5(2+\sqrt{5})\sqrt{Q_1}\},$$
$$= 80\{-25-47\sqrt{5}+(5(2-\sqrt{5})-5)\sqrt{Q}\},$$

that is,

$$A'B' \div \beta\gamma = 4\sqrt{5}\{-25-47\sqrt{5}+5(1-\sqrt{5})\sqrt{Q}\}$$
$$= -20(47+5\sqrt{5})+20\sqrt{5}(1-\sqrt{5})\sqrt{Q}, = A'';$$

and similarly we verify the values of B'', C'' and D''.

We have next

$$A'A'' = 160\sqrt{5}\{(10+5\sqrt{5}+\sqrt{Q})(18+13\sqrt{5}+\sqrt{5}\sqrt{Q})\},$$

or observing that $Q\sqrt{5}$ is $= 235(2+\sqrt{5})$, the whole term in $\{\ \}$ is

$$= (505+220\sqrt{5})+(470+235\sqrt{5})+(18+13\sqrt{5}+25+10\sqrt{5})\sqrt{Q},$$
$$= 975+455\sqrt{5}+(43+23\sqrt{5})\sqrt{Q} = 65\sqrt{5}(7+3\sqrt{5})+(43+23\sqrt{5})\sqrt{Q};$$

or we have

$$A'A'' = 160\sqrt{5}\{65\sqrt{5}(7+3\sqrt{5})+(43+23\sqrt{5})\sqrt{Q}\},$$
$$= 160\sqrt{5}(1+\sqrt{5})\{65\sqrt{5}(2+\sqrt{5})+(18+5\sqrt{5})\sqrt{Q}\},$$

and consequently

$$A'A'' \div \beta\gamma = 40(1+\sqrt{5})\{65\sqrt{5}(2+\sqrt{5})+(18+5\sqrt{5})\sqrt{Q}\}, = A;$$

and similarly we verify the values of B, C, D.

In the proposed equation $x^5+3000x^2+20000x-100000=0$, we have $r=3000$, $s=20000$, $t=-100000$, $\alpha\delta=-4\sqrt{5}$; the two equations to be verified thus are

$$A+B+C+D+40\sqrt{5}(A'+D'-B'-C')-100000=0,$$

and

$$5A''+15B''+10C''-10D''-\frac{\sqrt{5}}{4}(C'^2+2D'^2)$$
$$-150\sqrt{5}(C'+2D')-1600+20000=0.$$

As to the first of these, we have $A+B+C+D=156000$, $A'+D'-B'-C'$ $=-280\sqrt{5}$, and the equation thus is

$$156000+40\sqrt{5}(-280\sqrt{5})-10000=0,$$

which is right.

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For the second equation, if in the calculation we keep the radicals in the first instance distinct, we have

$$\begin{aligned}
5A'' + 15B'' + 10C'' - 10D'' &= -18800 + 3000\sqrt{5} \\
&\quad + (-1500 + 300\sqrt{5})\sqrt{Q} + (500 + 100\sqrt{5})\sqrt{Q_1}, \\
-150\sqrt{5}\,(C' + 2D') &= \{-450 - 70\sqrt{5} \\
&\quad + (20 + 4 + \sqrt{5})\sqrt{Q} + (-10 + 2\sqrt{5})\sqrt{Q_1}\}(-150\sqrt{5}) \\
-1600 + 20000 &= 18400 \\
-\frac{\sqrt{5}}{4}\,(C'^2 + 2D'^2) &= -\frac{\sqrt{5}}{4}\{282000 + 416800\sqrt{5} \\
&\quad + (-8800 - 4000\sqrt{5})\sqrt{Q} + (4400 - 2000\sqrt{5})\sqrt{Q_1}\}.
\end{aligned}$$

Substituting in the equation, we ought to have

$$\begin{aligned}
0 = &- 18800 + 3000\sqrt{5} + (-1500 + 300\sqrt{5})\sqrt{Q} + (500 + 100\sqrt{5})\sqrt{Q_1} \\
&+ 52500 + 67500\sqrt{5} + (-3000 - 3000\sqrt{5})\sqrt{Q} + (-1500 + 1500\sqrt{5})\sqrt{Q_1} \\
&+ 18400 \\
&- 52100 - 70500\sqrt{5} + (5000 + 2200\sqrt{5})\sqrt{Q} + (2500 - 1100\sqrt{5})\sqrt{Q_1}
\end{aligned}$$

that is,

$$0 = (500 - 500\sqrt{5})\sqrt{Q} + (1500 + 500\sqrt{5})\sqrt{Q_1},$$

viz.

$$0 = 500\,\{(1 - \sqrt{5})\sqrt{Q} + (3 + \sqrt{5})\sqrt{Q_1}\},$$

which is satisfied in virtue of $\sqrt{Q} = (2 + \sqrt{5})\sqrt{Q_1}$: this completes the verification.

*On the Theory of Substitution-Groups and its Applications to Algebraic Equations.**

By Oskar Bolza.

The object of the following paper (which is mostly a reproduction of a course of lectures which I had the honor of delivering at the Johns Hopkins University during the months of January and February, 1889) is to give an elementary introduction to the theory of substitution-groups and its application to Galois' theory of algebraic equations.

The paper is divided into two parts: *the first* develops the fundamental propositions on substitution-groups in connection with the theory of asymmetric functions of n indeterminate quantities, and concludes with a short sketch of the extension of the theory to groups of operations in general.

The *second part* deals with Galois' theory of algebraic equations, in particular their solution by radicals.

The bulk of the material is taken from: Jordan, Traité des Substitutions; Serret, Cours d'algèbre supérieure, and Netto, Substitutionen-Theorie.†

Besides, I have chiefly consulted

a) for the first part:‡ Klein, Vorlesungen über das Ikosaeder; Capelli, Sopra l'isomorfismo dei gruppi di sostituzioni, Giornale di Matematiche, Vol. 16; Dyck, Gruppentheoretische Studien, Math. Annalen, Bd. 22.

b) for the second part: Kronecker, Entwickelungen aus der Theorie der algebraischen Gleichungen, Berliner Monatsb. 1879; König, Ueber rationale Functionen von n Elementen, etc., Math. Annalen, Bd. 14; Bachmann, Ueber Galois' Theorie der algebraischen Gleichungen, Math. Ann., Bd. 18.

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* This subject being one on which no separate work is found in the English language, Dr. Bolza's development of it is published here, in the belief that it will prove extremely helpful to all students of the subject, especially by supplementing and illustrating the more extended works of Jordan and Netto. —The Editor.

† These treatises are referred to, in the following, simply as Jordan, Serret, Netto.

‡ In places where the developments would have exactly coincided with Netto's or Serret's, I have confined myself to an analysis of the proofs, referring for the details to the originals.

Moreover, the whole of my paper has been strongly influenced by a course of lectures by Professor Klein on the same subject which he delivered in the University of Göttingen during the summer of 1886. And I take the greatest pleasure in expressing to him my warmest thanks for his kind permission to make use of these lectures for the present publication.

FIRST PART.

THEORY OF RATIONAL FUNCTIONS OF n INDETERMINATE QUANTITIES AND OF SUBSTITUTION-GROUPS.

§1.—*Introduction: Lagrange's Researches.*

1. The theory of substitutions had its origin in Lagrange's discovery* that in all the known solutions of cubic and biquadratic equations the auxiliary irrationalities which enter into the expressions for the roots *are rationally expressible in terms of the roots of the given equation.*

The roots of the *cubic equation*, which we may, without loss of generality, assume in the reduced form

$$x^3 + px + q = 0, \tag{1}$$

are given by Cardanus' formula,

$$\left.\begin{aligned} x_1 &= \sqrt[3]{-\frac{q}{2} + \sqrt{R}} + \sqrt[3]{-\frac{q}{2} - \sqrt{R}}, \\ x_2 &= \omega\sqrt[3]{-\frac{q}{2} + \sqrt{R}} + \omega^2\sqrt[3]{-\frac{q}{2} - \sqrt{R}}, \\ x_3 &= \omega^2\sqrt[3]{-\frac{q}{2} + \sqrt{R}} + \omega\sqrt[3]{-\frac{q}{2} - \sqrt{R}}, \end{aligned}\right\} \tag{2}$$

R denoting the expression

$$R = \frac{q^2}{4} + \frac{p^3}{27}$$

and ω being an imaginary cube root of unity.

In the expressions (2), the sign of the square root and the value of one of the

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*Réflexions sur la résolution algébrique des équations, Oeuvres, Vol. III, p. 205.

two cube roots may be chosen at will; the choice of the other cube root is then determined by the condition

$$\sqrt[3]{-\frac{q}{2}+\sqrt{R}}\cdot\sqrt[3]{-\frac{q}{2}-\sqrt{R}}=-\frac{p}{3}, \tag{3}$$

which can always be satisfied, since the cube of the product on the left-hand side is found to be $-\frac{p^3}{27}$ (compare Serret, No. 505).

The radicals which enter into the expressions (2) of the roots are, in fact, rationally expressible in terms of x_1, x_2, x_3.

For, by multiplying the three equations (2) by 1, ω^2, ω respectively and adding, we have, since $1+\omega+\omega^2=0$,

$$\sqrt[3]{-\frac{q}{2}+\sqrt{R}}=\frac{x_1+\omega^2x_2+\omega x_3}{3}; \tag{4}$$

by multiplying the same equations by 1, ω, ω^2 and adding, we obtain

$$\sqrt[3]{-\frac{q}{2}-\sqrt{R}}=\frac{x_1+\omega x_2+\omega^2x_3}{3}. \tag{5}$$

Cubing the last two equations and subtracting the second from the first, we obtain

$$\sqrt{R}=\frac{1}{2.3^3}\left[(x_1+\omega^2x_2+\omega x_3)^3-(x_1+\omega x_2+\omega^2x_3)^3\right],$$

or, after an easy simplification,

$$\sqrt{R}=\frac{\sqrt{-3}}{18}(x_2-x_3)(x_3-x_1)(x_1-x_2). \tag{6}$$

Thus our statement is verified.

2. On the properties of the expression

$$\psi_1=x_1+\omega x_2+\omega^2x_3,$$

to which the preceding analysis of Cardan's formula has led, Lagrange has founded a direct method of solving the complete cubic equation

$$x^3-c_1x^2+c_2x-c_3=0. \tag{7}$$

If, in the function ψ_1, the letters x_1, x_2, x_3 are interchanged among themselves in all possible ways, ψ_1 is changed successively into the following six functions (the six "*values*" of the function ψ_1):

$$\left.\begin{aligned}
\psi_1 &= x_1 + \omega x_2 + \omega^2 x_3, \\
\psi_2 &= x_2 + \omega x_3 + \omega^2 x_1 = \omega^2\psi_1, \\
\psi_3 &= x_3 + \omega x_1 + \omega^2 x_2 = \omega\psi_1, \\
\psi_4 &= x_1 + \omega x_3 + \omega^2 x_2, \\
\psi_5 &= x_3 + \omega x_2 + \omega^2 x_1 = \omega^2\psi_4, \\
\psi_6 &= x_2 + \omega x_1 + \omega^2 x_3 = \omega\psi_4,
\end{aligned}\right\} \tag{8}$$

These six "values" are the roots of the equation of the sixth degree,

$$(\psi - \psi_1)(\psi - \psi_2)(\psi - \psi_3)(\psi - \psi_4)(\psi - \psi_5)(\psi - \psi_6) = 0,$$

or, on account of (8),

$$\psi^6 - (\psi_1^3 + \psi_4^3)\,\psi^3 + \psi_1^3\psi_4^3 = 0,$$

whose coefficients are easily seen to be symmetric functions of x_1, x_2, x_3 and are therefore rationally expressible in terms of the coefficients c_1, c_2, c_3 of (7); the result is (see Serret, No. 508),

$$\begin{aligned}
\psi_1^3 + \psi_4^3 &= 2c_1^3 - 9c_1c_2 + 27c_3, \\
\psi_1\psi_4 &= c_1^2 - 3c_2.
\end{aligned} \tag{9}$$

Hence our equation becomes

$$\psi^6 - (2c_1^3 - 9c_1c_2 + 27c_3)\,\psi^3 + (c_1^2 - 3c_2)^3 = 0. \tag{10}$$

To solve it we put $\psi^3 = \theta$ and obtain for θ the quadratic equation

$$\theta^2 - (2c_1^3 - 9c_1c_2 + 27c_3)\,\theta + (c_1^2 - 3c_2)^3 = 0; \tag{11}$$

denoting by θ_1 and θ_2 its two roots, we have

$$\psi_1 = \sqrt[3]{\theta_1}, \quad \psi_4 = \sqrt[3]{\theta_2}.$$

On account of the relation (9), the cube roots must be chosen such that

$$\sqrt[3]{\theta_1}\sqrt[3]{\theta_2} = c_1^2 - 3c_2. \tag{12}$$

ψ_1 and ψ_4 being thus found, the roots x_1, x_2, x_3 can be obtained from the three linear equations

$$\begin{aligned}
x_1 + \omega x_2 + \omega^2 x_3 &= \sqrt[3]{\theta_1}, \\
x_1 + \omega^2 x_2 + \omega x_3 &= \sqrt[3]{\theta_2}, \\
x_1 + x_2 + x_3 &= c_1,
\end{aligned}$$

which give

$$\left.\begin{aligned} x_1 &= \frac{c_1 + \sqrt[3]{\theta_1} + \sqrt[3]{\theta_2}}{3}, \\ x_2 &= \frac{c_1 + \omega^2\sqrt[3]{\theta_1} + \omega\sqrt[3]{\theta_2}}{3}, \\ x_3 &= \frac{c_1 + \omega\sqrt[3]{\theta_1} + \omega^2\sqrt[3]{\theta_2}}{3}. \end{aligned}\right\} \tag{13}$$

3. The *biquadratic equation*

$$x^4 + qx^2 + rx + s = 0 \tag{1}$$

is solved, after Ferrari, by means of the cubic "resolvent equation"

$$\xi^3 - q\xi^2 - 4s\xi - (r^2 - 4qs) = 0. \tag{2}$$

If ξ_1 denote one of the roots of (2), the left-hand side of (1) may be written

$$\left(x^2 + \frac{\xi_1}{2}\right)^2 + (\xi_1 - q)\left[x - \frac{r}{2(\xi_1 - q)}\right]^2 = 0.$$

Hence two roots of (1), say x_1 and x_2, are found by solving the quadratic equation

$$x^2 + \sqrt{\xi_1 - q}\, x + \left(\frac{\xi_1}{2} - \frac{r}{2\sqrt{\xi_1 - q}}\right) = 0, \tag{3}$$

the two others, x_3 and x_4, by solving the quadratic equation

$$x^2 - \sqrt{\xi_1 - q}\, x + \left(\frac{\xi_1}{2} + \frac{r}{2\sqrt{\xi_1 - q}}\right) = 0. \tag{4}$$

Again the two auxiliary irrationalities ξ_1 and $\sqrt{\xi_1 - q}$ are rationally expressible in terms of the roots of (1); for from (3) follows:

$$x_1 + x_2 = -\sqrt{\xi_1 - q}, \quad x_1x_2 = \frac{\xi_1}{2} - \frac{r}{2\sqrt{\xi_1 - q}},$$

and from (4),

$$x_3 + x_4 = +\sqrt{\xi_1 - q}, \quad x_3x_4 = \frac{\xi_1}{2} + \frac{r}{2\sqrt{\xi_1 - q}}.$$

Hence

$$\left.\begin{aligned} \xi_1 &= x_1x_2 + x_3x_4, \\ \sqrt{\xi_1 - q} &= \frac{x_3 + x_4 - x_1 - x_2}{2}. \end{aligned}\right\} \tag{5}$$

4. From his analysis of Ferrari's solution, Lagrange derived the following method of solving the biquadratic equation

$$x^4 - c_1x^3 + c_2x^2 - c_3x + c_4 = 0. \tag{6}$$

The function $x_1x_2+x_3x_4$ is a "three-valued function of the roots"; for it takes, on the whole, three *different* values, if the letters x_1, x_2, x_3, x_4 are interchanged among themselves in all possible ways, viz.

$$\xi_1=x_1x_2+x_3x_4,\quad \xi_2=x_1x_3+x_2x_4,\quad \xi_3=x_1x_4+x_2x_3. \tag{7}$$

These three expressions are the roots of the cubic equation

$$(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)=0,$$

whose coefficients are immediately seen to be symmetric functions of x_1, x_2, x_3, x_4 and are therefore rationally expressible in terms of the coefficients c_1, c_2, c_3, c_4 of (6); the result is (see Serret, No. 515):

$$\xi^3-c_2\xi^2+(c_1c_3-c_4)\xi-(c_1^2c_4-4c_2c_4+c_3^2)=0. \tag{8}$$

ξ_1 being found by solving this equation, the roots x_1, x_2, x_3, x_4 can be obtained as follows:

From the two equations

$$x_1x_2+x_3x_4=\xi_1,\quad x_1x_2\,.\,x_3x_4=c_4$$

it follows that the two quantities x_1x_2 and x_3x_4 are the roots of the quadratic equation

$$\xi^2-\xi_1\xi+c_4=0. \tag{9}$$

Further, x_1+x_2 and x_3+x_4 are expressible in terms of x_1x_2 and x_3x_4 by means of the two equations

$$\begin{aligned}(x_1+x_2)+\qquad\quad (x_3+x_4)&=c_1,\\ x_3x_4(x_1+x_2)+x_1x_2(x_3+x_4)&=c_3.\end{aligned} \tag{10}$$

But x_1+x_2 and x_1x_2 on the one hand, and x_3+x_4 and x_3x_4 on the other hand being found, we can construct the two quadratic equations whose roots are x_1 and x_2 and x_3 and x_4 respectively (for the details see Serret, No. 515).

5. Lagrange's results on the cubic and biquadratic equations suggested the idea of solving, in a similar manner, equations of a higher degree with the aid of rational functions of the roots; thus Lagrange was led to study generally the properties of rational functions of the roots $x_1, x_2 \ldots . x_n$ of an equation of the n^{th} degree with respect to the changes which they undergo when the letters $x_1, x_2 \ldots . x_n$ are interchanged among themselves, and he established the following fundamental theorems:

1). The number, say ρ, of different values which a rational function of the roots takes, if the letters $x_1, x_2 \ldots . x_n$ are interchanged among themselves in all possible ways, is always a divisor of $n!$

2). The ρ different values of a ρ-valued function of the roots satisfy an algebraic equation of the ρ^{th} degree whose coefficients are rational functions of the coefficients of the given equation.

3). If a rational function $\phi(x_1, x_2 \ldots . x_n)$ remains unaltered by all those permutations of the letters $x_1, x_2 \ldots . x_n$ which leave unaltered another function $\psi(x_1, x_2 \ldots . x_n)$, then ϕ is rationally expressible in terms of ψ and of the coefficients of the given equation.

Whereas Lagrange directed his attention chiefly to the different values of a rational function of the roots, an essential progress in the theory was made by Cauchy,* who first systematically considered those systems of permutations—or, as he says, substitutions—of the roots which leave a given rational function unaltered; and in doing so, Cauchy has become the founder of an independent Theory of Substitutions, which at the same time throws a new light on Lagrange's discoveries.

We shall begin with an exposition of the elements of Cauchy's theory and give the proofs of Lagrange's theorems in their proper places.

§2.—*Elementary Propositions on Substitutions.*

6. Cauchy distinguishes between a permutation and the operation of permuting a number of objects; he calls this operation a *substitution.* The substitution which replaces x_1 by x_α, x_2 by x_β, ... x_n by x_λ, the indices $\alpha, \beta, \ldots . \lambda$ being some permutation of the numbers $1, 2, \ldots . n$, is usually denoted by

$$\begin{pmatrix} x_1 & x_2 & \ldots . & x_n \\ x_\alpha & x_\beta & \ldots . & x_\lambda \end{pmatrix};$$

sometimes, however, it is more convenient to write the letters of the first line, and accordingly also those of the second, in a different order; for instance,

$$\begin{pmatrix} x_2 & x_n & \ldots . & x_1 \\ x_\beta & x_\lambda & \ldots . & x_\alpha \end{pmatrix} \text{ or } \begin{pmatrix} x_n & x_2 & \ldots . & x_1 \\ x_\lambda & x_\beta & \ldots . & x_\alpha \end{pmatrix}, \text{ etc.}$$

There exist evidently as many different substitutions between n letters as there exist permutations, viz. $n!$; among them is included the "identical substitution"

$$\begin{pmatrix} x_1 & x_2 & \ldots . & x_n \\ x_1 & x_2 & \ldots . & x_n \end{pmatrix},$$

which leaves all the letters unaltered and which is denoted by 1.

* Exercises d'analyse et de physique mathématique.

7. If, in a rational function $\phi(x_1, x_2 \ldots . x_n)$ of $x_1, x_2 \ldots . x_n$, we apply to the letters $x_1, x_2 \ldots . x_n$ the substitution

$$a = \begin{pmatrix} x_1 \, x_2 \ldots . x_n \\ x_\alpha \, x_\beta \ldots . x_\lambda \end{pmatrix},$$

the function ϕ is changed into $\phi(x_\alpha, x_\beta \ldots . x_\lambda)$; we shall denote, with Jordan, this function by

$$\phi_a(x_1, x_2 \ldots . x_n).$$

Corresponding to the $n!$ substitutions which we denote by

$$1, a, b, c, \ldots . , \tag{1}$$

we thus obtain $n!$ functions

$$\phi, \phi_a, \phi_b, \phi_c, \ldots . \tag{2}$$

Several cases may present themselves:

1). The $n!$ functions (2) may all be distinct; such an "$n!$-valued function" is for instance, as is easily verified,

$$\phi = m_1 x_1 + m_2 x_2 + \ldots . + m_n x_n,$$

provided the constant factors $m_1, m_2, \ldots . m_n$ be all distinct (see Serret, No. 491; Netto, §29).

2). The $n!$ functions (2) may all be equal; that is, ϕ may "remain unaltered" by all the $n!$ substitutions. It is then called a "*one-valued*" or *symmetric function.*

3). These are the two extreme cases; in general, the function ϕ remains unaltered by some of the $n!$ substitutions and is changed by others. If there are among the functions (2) ρ *different* functions, ϕ is called a ρ-*valued function.*

To complete these definitions, we must add that we consider throughout the first part of this memoir the letters $x_1, x_2, \ldots . x_n$ as *indeterminate quantities,* and accordingly two rational functions of $x_1, x_2, \ldots . x_n$ as equal only when they are identical for all sets of values of the letters $x_1, x_2, \ldots . x_n$ (compare below, No. 55).

8. *Example* I. With *three letters* x_1, x_2, x_3 we have $3! = 6$ substitutions; they are of the following "types":*

1). The identical substitution 1.

*Substitutions of the same type differ only by the notation of their letters, and are called *similar substitutions* (compare below, No. 33).

2). Each of the substitutions

$$c = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_1 & x_3 & x_2 \end{pmatrix}, \quad d = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_3 & x_2 & x_1 \end{pmatrix}, \quad e = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_1 & x_3 \end{pmatrix}$$

leaves unaltered one letter and interchanges the two others; generally substitutions which only interchange two of the n letters, say x_α and x_β, are called *transpositions* and are usually written in the abbreviated notation $(x_\alpha x_\beta)$ or $(x_\beta x_\alpha)$; so here

$$c = (x_2 x_3), \quad d = (x_1 x_3), \quad e = (x_1 x_2).$$

3). The substitutions

$$a = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_1 \end{pmatrix}, \quad b = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \end{pmatrix}$$

interchange the three letters cyclically; they are called *circular substitutions* and are usually denoted by

$$a = (x_1 x_2 x_3) \text{ or } (x_2 x_3 x_1) \text{ or } (x_3 x_1 x_2),$$
$$b = (x_1 x_3 x_2) \text{ or } (x_2 x_1 x_3) \text{ or } (x_3 x_2 x_1),$$

the symbol $(x_\alpha x_\beta x_\gamma \ldots . x_\kappa x_\lambda)$ denoting generally the "circular" substitution which replaces x_α by x_β, x_β by $x_\gamma \ldots . x_\kappa$ by x_λ, x_λ by x_α. (Circular substitution of two letters = transposition.)

Applying these substitutions to the function

$$\theta = (x_1 + \omega x_2 + \omega^2 x_3)^3$$

we find (compare No. 2)

$$\theta_1 = \theta_a = \theta_b = (x_1 + \omega x_2 + \omega^2 x_3)^3,$$
$$\theta_c = \theta_d = \theta_e = (x_1 + \omega^2 x_2 + \omega x_3)^3;$$

hence θ is a two-valued function.

For the function

$$\phi = (x_2 - x_3)(x_3 - x_1)(x_1 - x_2)$$

we have

$$\phi_1 = \phi_a = \phi_b = \phi,$$
$$\phi_c = \phi_d = \phi_e = -\phi;$$

thus ϕ is a two-valued and, in particular, an *alternate function*, since its two values differ only in sign.

9. *Example* II. Between four letters x_1, x_2, x_3, x_4 there exist $4! = 24$ substitutions; we may distinguish the following types:

a). The *substitution* 1.

b). $\frac{4.3}{1.2} = 6$ *transpositions*; they are, if we write for shortness, only the indices $(1\,2)$; $(1\,3)$; $(1\,4)$; $(2\,3)$; $(2\,4)$; $(3\,4)$.

c). $4.2 = 8$ *circular substitutions of three letters*, viz.

$$(2\,3\,4);\ (2\,4\,3);\ (1\,3\,4);\ (1\,4\,3);\ (1\,2\,4);\ (1\,4\,2);\ (1\,2\,3);\ (1\,3\,2).$$

d). $3! = 6$ *circular substitutions of four letters*, viz.

$$(1\,2\,3\,4);\ (1\,2\,4\,3);\ (1\,3\,2\,4);\ (1\,3\,4\,2);\ (1\,4\,2\,3);\ (1\,4\,3\,2).$$

e). Finally, the three substitutions

$$\begin{pmatrix}1\,2\,3\,4\\2\,1\,4\,3\end{pmatrix};\ \begin{pmatrix}1\,2\,3\,4\\3\,4\,1\,2\end{pmatrix};\ \begin{pmatrix}1\,2\,3\,4\\4\,3\,2\,1\end{pmatrix};$$

they are usually written in the abbreviated notation

$$(1\,2)(3\,4);\ (1\,3)(2\,4);\ (1\,4)(2\,3);$$

that is, the first substitution interchanges the two letters x_1 and x_2 on the one hand and x_3 and x_4 on the other hand, etc.

Similarly, every substitution between any number of letters can be "decomposed into cycles" of different letters; for instance,

$$\begin{pmatrix}1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\\3\ 9\ 6\ 4\ 10\ 7\ 1\ 5\ 2\ 8\end{pmatrix} = (1,\ 3,\ 6,\ 7)(2,\ 9)(5,\ 10,\ 8);$$

the letter x_4, which is not altered, may either be suppressed or added as a cycle of one letter (see Serret, No. 408; Netto, §22).

Applying these substitutions to the function

$$\xi_1 = x_1x_2 + x_3x_4$$

of No. 4 we obtain the following table:

$x_1x_2 + x_3x_4$	1;	(1 2);	(3 4);	(1 2)(3 4);	(1 3)(2 4);	(1 4)(2 3);	(1 3 2 4);	(1 4 2 3)
$x_1x_3 + x_2x_4$	(2 3 4);	(1 3 4 2);	(2 3);	(1 3 2);	(1 4 3);	(1 2 4);	(1 4);	(1 2 4 3)
$x_1x_4 + x_2x_3$	(2 4 3);	(1 4 3 2);	(2 4);	(1 4 2);	(1 2 3);	(1 3 4);	(1 2 3 4);	(1 3)

the substitutions of the first line leave $\xi_1 = x_1x_2 + x_3x_4$ unaltered, those of the second line change it into $\xi_2 = x_1x_3 + x_2x_4$, those of the third into $\xi_3 = x_1x_4 + x_2x_3$.

10. If we apply to a rational function $\phi(x_1, x_2 \ldots . x_n)$ first a substitution

$$a = \begin{pmatrix}x_1\ x_2 \ldots . x_n\\x_\alpha\ x_\beta \ldots . x_\lambda\end{pmatrix},$$

ϕ is changed into $\phi_a = \phi(x_\alpha, x \ \ldots . x_\lambda)$; if, now, we apply to this new function ϕ_a another substitution b, which replaces x_α by $x_{\alpha'}$, x_β by $x_{\beta'}$, ... x_λ by $x_{\lambda'}$, and which consequently may be written

$$b = \begin{pmatrix} x_\alpha & x_\beta & \ldots . & x_\lambda \\ x_{\alpha'} & x_{\beta'} & \ldots . & x_{\lambda'} \end{pmatrix},$$

ϕ_a is changed into $(\phi_a)_b = \phi(x_{\alpha'}, x_{\beta'}, \ldots . x_{\lambda'})$. But the permutation $x_{\alpha'} x_{\beta'} \ldots x_{\lambda'}$ could have been derived directly from the original permutation $x_1 x_2 \ldots . x_n$ by the substitution

$$\begin{pmatrix} x_1 & x_2 & \ldots . & x_n \\ x_{\alpha'} & x_{\beta'} & \ldots . & x_{\lambda'} \end{pmatrix}.$$

This substitution, which is equivalent to the successive application of—first a and then b, is called the *product of a by b* and is denoted* by ab.

Examples: 1).

$$\begin{aligned} a &= (x_1\, x_2\, x_3) = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_1 \end{pmatrix}, \\ c &= (x_2\, x_3) \quad = \begin{pmatrix} x_2 & x_3 & x_1 \\ x_3 & x_2 & x_1 \end{pmatrix}, \\ ac &= \qquad\qquad = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_3 & x_2 & x_1 \end{pmatrix} = (x_1\, x_3), \\ ca &= \qquad\qquad\qquad\qquad\quad (x_1\, x_2). \end{aligned}$$

2).

$$a = (x_1\, x_2)(x_3\, x_4); \quad b = (x_1\, x_3)(x_2\, x_4),$$
$$ab = (x_1\, x_4)(x_2\, x_3); \quad ba = (x_1\, x_4)(x_2\, x_3).$$

11. *Elementary propositions on products of substitutions* (see Serret, No. 405; Netto, §27):

a). *The commutative law does not, in general, hold* for products of substitutions; in special cases, however, it may happen that $ba = ab$ (see the above examples). Two such substitutions are called *interchangeable* ("échangeables," Jordan, Serret).

Two substitutions which operate upon different letters are always interchangeable:

$$(x_1\, x_2\, x_3)(x_4\, x_5) = (x_4\, x_5)(x_1\, x_2\, x_3).$$

b). The *substitution* 1 may be suppressed in a product since $a.1 = 1.a = a$; conversely, if $ba = a$ or $ab = a$, then b must be 1.

* We adopt the notation of Jordan and Netto on account of its practical advantages in applications to rational functions. Serret writes ba, in accordance with the usual notation in arithmetic and quaternions.

c). To every substitution a belongs another substitution a' called its *inverse*, and denoted by a^{-1}, such that

$$aa' = a'a = 1, \text{ viz. if}$$

$$a = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ x_\alpha & x_\beta & \dots & x_\lambda \end{pmatrix}, \text{ then evidently}$$

$$a^{-1} = \begin{pmatrix} x_\alpha & x_\beta & \dots & x_\lambda \\ x_1 & x_2 & \dots & x_n \end{pmatrix}.$$

d). A *product of three substitutions* is defined by $abc = (ab)c$; *the associative law holds*

$$(ab)c = a(bc).$$

12. A product of equal substitutions is called a *power* and denoted accordingly: $a.a = a^2$, $a.a.a = a^3$, etc. On account of the associative law we have $a^\alpha . a^\beta = a^{\alpha+\beta}$.

If we form the successive powers of any substitution a:

$$a, a^2, a^3 \dots \textit{in infinitum},$$

some of the powers must be equal, since there exist only a finite number of different substitutions between n letters. Hence it is easy to infer that some power of a must be equal to 1 (Serret, No. 406); if a^μ be the lowest power of a which is equal to 1,

$$a^\mu = 1,$$

then μ is called the *order* or *period* of the substitution a.

The first μ powers,

$$a, a^2, \dots a^\mu,$$

are all distinct, and are periodically repeated if we continue the series of powers beyond a^μ: $\quad a^{\mu+1} = a;\ a^{\mu+2} = a^2 \dots;$

generally, $\quad a^{\alpha\mu+\beta} = a^\beta.$

It is convenient to introduce also *negative powers*; $a^{-\beta}$ is defined by $a^{\alpha\mu-\beta}$, the integer α being chosen such that $0 < \alpha\mu - \beta \overline{\gtrless} \mu$; in particular we have

$$a^{-1} = a^{\mu-1}. \qquad \text{(Compare No. 11, } c.)$$

Further, a^0 is defined to be $= 1$.

Examples:

1). $a = (2\,3)$; $a^2 = 1$; order 2.
2). $a = (1\,2\,3)$, $a^2 = (1\,3\,2)$, $a^3 = 1$; order 3.
3). $a = (1\,2\,3\,4)$, $a^2 = (1\,3)(2\,4)$, $a^3 = (1\,4\,3\,2)$, $a^4 = 1$; order 4.

Generally, *the order of a circular substitution of m letters is m* (Serret, No. 408).

4). $a=(1\,2)(3\,4)$, $a^2=(1\,2)(3\,4).(1\,2)(3\,4)=(1\,2)^2(3\,4)^2=1$; order 2.

5). $a=(1\,2\,3)(4\,5)$, $a^2=(1\,3\,2)$, $a^3=(4\,5)$,
$a^4=(1\,2\,3)$, $a^5=(1\,3\,2)(4\,5)$, $a^6=1$; order 6.

Generally, *the order of any substitution is the least common multiple of the orders of its cycles* (Serret, No. 409). Hence the corollary: A substitution of order p among a *prime* number p of letters must necessarily be a circular substitution.

§3.—*Rational Functions and Substitution-Groups.*

13. Let us select among the $n!$ substitutions between n letters $x_1, x_2 \ldots . x_n$ all those substitutions which leave a given rational function $\phi(x_1, x_2 \ldots . x_n)$ unaltered; and let a be any one of them, so that $\phi_a=\phi$. This equation is an identity, holding for all sets of values of the letters $x_1, x_2 \ldots . x_n$ (No. 7), and remains therefore true if operated upon by any substitution whatever, say b; that is, according to the notations of No. 10: $\phi_{ab}=\phi_b$.

In particular, if b be itself one of those substitutions which leave ϕ unaltered (including the case $b=a$), we have $\phi_b=\phi$, and consequently also $\phi_{ab}=\phi$. Hence also the product ab is one of those substitutions which leave ϕ unaltered.

Now, a system of substitutions which possesses this characteristic property, that the product of any two of its substitutions—equal or different—belongs itself to the system, is called a *group of substitutions.**

Using this definition, we have the theorem:

The totality of those substitutions which leave a rational function unaltered, constitute a group.

Evidently every product of *any number* of substitutions of a group belongs again to the group.

The number of different substitutions of a group is called its *order;* the number of letters which are operated upon, its *degree.*

We shall call the group G of those substitutions which leave the function ϕ unaltered, the *group of the function* ϕ, and say the function ϕ belongs to the group G.

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*Cauchy uses the term "système de substitutions conjuguées"; Galois introduced the term "groupe," which is now generally adopted.

14. *Example* I. The rational functions of x_1, x_2, x_3 which occur in the solution of the cubic equation (No. 1, 2) furnish the following *substitution-groups between three letters:*

a). The system of all the six—in the general case $n!$— substitutions constitute a group, G_6—in the general case $G_{n!}$; it is the group of the symmetric functions, and is called the *symmetric group.*

b). The alternate function

$$\phi = (x_2 - x_3)(x_3 - x_1)(x_1 - x_2)$$

belongs to the group (see No. 8)

$$G_3 = [1;\ (x_1 x_2 x_3);\ (x_1 x_3 x_2)],$$

called the *alternate group.*

c). The three functions x_1, x_2, x_3 belong to the three groups

$$G_2' = [1; (x_2 x_3)];\quad G_2'' = [1; (x_1, x_3)];\quad G_2''' = [1; (x_1 x_2)]$$

respectively.

d). Finally the substitution 1 may be considered to constitute by itself a group G_1, since $1.1 = 1$; this is the group of any function such as $x_1 + \omega x_2 + \omega^2 x_3$ which changes its value by every substitution except unity.

In order to show that the systems of substitutions just enumerated constitute, in fact, groups, we add the "multiplication table" for the six substitutions (notations of No. 8):

	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	b	1	d	e	c
b	b	1	a	e	c	d
c	c	e	d	1	b	a
d	d	c	e	a	1	b
e	e	d	c	b	a	1

the table gives, for instance, the product $ac = d$ in the intersection of the line a and the column c.

15. *Example* II. The rational functions which occur in the solution of the biquadratic equation (No. 3, 4) furnish the following *substitution-groups between four letters*:

a). The symmetric group G_{24}, consisting of all the 24 substitutions.

b). The function $\xi_1 = x_1 x_2 + x_3 x_4$ belongs to the group (see No. 9)

$$G_8 = [1; (1\,2); (3\,4); (1\,2)(3\,4); (1\,3)(2\,4); (1\,4)(2\,3); (1\,3\,2\,4); (1\,4\,2\,3)].$$

Since $\xi_2 = x_1 x_3 + x_2 x_4$ can be derived from ξ_1 by interchanging the two letters x_2 and x_3, the group of ξ_2 is evidently obtained by interchanging everywhere within the substitutions of G_8 the letters x_2 and x_3; thus we find

$$G_8' = [1; (1\,3); (2\,4); (1\,3)(2\,4); (1\,2)(3\,4); (1\,4)(3\,2); (1\,2\,3\,4); (1\,4\,3\,2)];$$

similarly the group of $\xi_3 = x_1 x_4 + x_2 x_3$ is found by interchanging within the substitutions of G_8 the letters x_2 and x_4:

$$G_8'' = [1; (1\,4); (3\,2); (1\,4)(3\,2); (1\,3)(4\,2); (1\,2)(4\,3); (1\,3\,4\,2); (1\,2\,4\,3)].$$

Such groups as G_8, G_8', G_8'' which differ only in the letters on which they operate, are called *similar groups* (compare below No. 33).

c). The function $$x_1 + x_2 - x_3 - x_4$$
belongs to the group
$$H_4 = [1; (1\,2); (3\,4); (1\,2)(3\,4)]$$
of order 4.

All the substitutions of this group are at the same time contained in the group G_8; hence H_4 is called a subgroup of G_8. Generally a group H is called a *subgroup* of another group G if all the substitutions of H are contained in G, including the extreme case $H = G$.

16. To find further interesting substitution-groups between four letters, we remember that, in solving the cubic resolvent equation (8) of No. 4, we have to pass, according to No. 2, through the rational functions of ξ_1, ξ_2, ξ_3:

$$\phi = (\xi_2 - \xi_3)(\xi_3 - \xi_1)(\xi_1 - \xi_2)$$

and

$$\psi = \xi_1 + \omega\xi_2 + \omega^2\xi_3.$$

Replacing ξ_1, ξ_2, ξ_3 by their expressions in terms of x_1, x_2, x_3, we obtain

$$\xi_2 - \xi_3 = (x_1 - x_2)(x_3 - x_4),$$
$$\xi_3 - \xi_1 = (x_1 - x_3)(x_4 - x_2),$$
$$\xi_1 - \xi_2 = (x_1 - x_4)(x_2 - x_3);$$

hence ϕ is the product of all the six differences of the letters x_1, x_2, x_3, x_4:

$$\phi = \Pi_{(\alpha, \beta)} (x_\alpha - x_\beta).$$

The function ϕ changes its sign if operated upon by any transposition. Now *any substitution can easily be decomposed into a product of transpositions* (Serret, No. 422); and if the product contains an even number of transpositions, the given substitution will leave ϕ unaltered; if an odd number, it will change ϕ into $-\phi$; ϕ is therefore an "alternate function"; it is even the *simplest alternate function.*

Further, it follows that, though a given substitution may be decomposed in various ways into a product of transpositions, all these different decompositions must contain either all an even number of transpositions or all an odd number, according as the given substitution does not or does change the alternate function ϕ. In the former case the substitution is called *even* or positive, in the latter, *odd* or negative.

To determine whether a given substitution is even or odd, it is sufficient to decompose it into cycles of different letters and to remark that a circular substitution of μ letters is odd when μ is even, and even when μ is odd, since

$$(x_1, x_2, x_3 \ldots . x_\mu) = (x_1 x_2)(x_1 x_3)(x_1 x_4) \ldots . (x_1 x_\mu).$$

Hence a substitution is even or odd according as the number of letters on which it operates *minus* the number of its cycles is even or odd.

The even substitutions between four letters are therefore: 1;
the three substitutions

$$(1\,2)(3\,4);\ (1\,3)(2\,4);\ (1\,4)(2\,3);$$

and the eight circular substitutions of three letters.

These twelve substitutions form a group G_{12}, the group of the alternate function $\Pi(x_\alpha - x_\beta)$, called the *alternate group* of four letters.

Analogous results hold for n letters.

The product of all the $\frac{n(n-1)}{2}$ differences between n letters remains unaltered by the $\frac{n!}{2}$ even substitutions which constitute the alternate group $G_{\frac{n!}{2}}$, and changes its sign by the $\frac{n!}{2}$ odd substitutions (see Serret, No. 429; Netto, §35).

The square of the simplest alternate function $\Pi(x_\alpha - x_\beta)$ is a symmetric function, known under the name of the *discriminant* of the equation whose roots $x_1, x_2 \ldots x_n$ are; it is usually denoted by Δ.

17. The function

$$\psi = \xi_1 + \omega\xi_2 + \omega^2\xi_3 = (x_1x_2 + x_3x_4) + \omega(x_1x_3 + x_2x_4) + \omega^2(x_1x_4 + x_2x_3)$$

remains unaltered by those substitutions which leave ξ_1, ξ_2 and ξ_3 simultaneously unaltered, and by no other; that is, by those substitutions which are common to the three groups G_8, G_8', G_8'' of No. 15; they are

$$1;\ a = (12)(34);\ b = (13)(24);\ c = (14)(23);$$

they form in fact a group G_4, since

$$\begin{gathered} a^2 = 1, \qquad b^2 = 1, \qquad c^2 = 1, \\ ab = ba = c,\ bc = cb = a,\ ca = ac = b. \end{gathered}$$

This group is called the *four-group* ("Vierergruppe," Klein); it has the remarkable property that any two of its substitutions are interchangeable.

From the definition of a group it follows that generally those substitutions which are common to two groups G and H constitute by themselves a group, the *greatest common subgroup* of G and H. If ϕ and ψ are two functions which belong to the groups G and H respectively, then the function

$$\chi = a\phi + b\psi,$$

where a and b are two constant parameters, belongs to the greatest common subgroup of G and H (Netto, §44).

18. Every rational function of $x_1, x_2 \ldots . x_n$ belongs, according to No. 13, to a certain substitution-group; conversely, *any substitution-group G between n letters $x_1, x_2 \ldots . x_n$ being given, it is always possible to construct rational functions of $x_1, x_2 \ldots . x_n$ which remain unaltered by the substitutions of G and by no other substitutions.*

Proof: Let
$$G = [a, b, \ldots . l] \tag{1}$$

be the given group; we first construct, according to No. 7, an $n!$-valued function, as
$$V = m_1x_1 + m_2x_2 + \ldots . m_nx_n$$

and apply to V all the substitutions of G; we thus obtain the functions

$$V_a,\ V_b, \ldots . V_l, \tag{2}$$

which are all distinct.

If, now, we apply to these functions simultaneously any substitution c of G, they are changed into

$$V_{ac},\ V_{bc},\ \ldots.\ V_{lc}. \tag{3}$$

These values are a permutation of the values (2); for the substitutions

$$ac,\ bc,\ \ldots.\ lc$$

belong all to the group G according to the definition of a group; moreover, they are all distinct, since, for instance, from $ac = bc$ we should deduce by multiplying by c^{-1}: $a = b$.

Hence any symmetric function of the values (2),

$$\text{Sym.}\,(V_a,\ V_b\ \ldots.\ V_l) = \phi\,(x_1,\ x_2\ \ldots.\ x_n)$$

remains unaltered by all the substitutions of G, and, in general, by no others. Thus for instance the product

$$V_a V_b\ \ldots.\ V_l$$

belongs to the group G, provided the coefficients $m_1,\ m_2\ \ldots.\ m_n$ be properly chosen.

Example: $n = 3,\ G = [1;\ a = (x_1 x_2 x_3);\ b = (x_1 x_3 x_2)]$.

Choosing for the coefficients m the values

$$m_1 = 1,\ m_2 = \omega,\ m_3 = \omega^2 \qquad (\omega^3 = 1),$$

we have $\quad V = x_1 + \omega x_2 + \omega^2 x_3,$

$$V_a = \omega^2 V,\ \ V_b = \omega V,$$

hence $\quad \phi\,(x_1,\ x_2,\ x_3) = V_1 V_a V_b = (x_1 + \omega x_2 + \omega^2 x_3)^3$

belongs to the group G (compare No. 8).

19. Thus a perfect reciprocity between rational functions and substitution groups is established: Every rational function belongs to a certain group and to every group belongs an infinity of rational functions.

All the rational functions of $x_1,\ x_2\ \ldots.\ x_n$ may therefore conveniently be classified according to the groups to which they belong; and thus the problem arises: *To determine independently all the substitution-groups which can be formed with n letters.*

To solve this problem we may proceed as follows:

We write down all the $n!$ substitutions and select among them, at random, any combination of m substitutions,

$$a,\ b,\ \ldots.\ k \tag{1}$$

and construct the multiplication-table containing the m^2 products

$$\begin{array}{l} a^2,\ ab \ldots\ldots ak, \\ ba,\ b^2 \ \ldots\ldots bk, \\ \ldots\ldots\ldots\ldots \\ ka,\ kb \ldots\ldots k^2. \end{array}$$

If this table contains no other substitutions than those of (1), the substitutions (1) constitute by themselves a group. If, on the contrary, it contains still other substitutions,

$$l,\ m,\ldots p, \tag{2}$$

not included in (1), then the substitutions (1) do not constitute a group. In this case we add the new substitutions (2) to the system (1) and form the multiplication-table for the enlarged system. Thus we continue until at last we arrive at a system of substitutions which do constitute a group, and we must finally arrive at such a system since there exist only a finite number $n!$ of different substitutions. We denote this group by $G(a,\ b \ldots\ldots k)$, it is said to be *generated* by the substitutions $a,\ b \ldots k$; it is *the smallest group which contains the substitutions* $a,\ b \ldots\ldots k$; that is to say, every group which contains these substitutions contains necessarily all the substitutions of $G(a,\ b \ldots\ldots k)$.

Applying this proceeding to all the possible combinations of one, two, three ...., etc. substitutions, we obtain all the possible substitution-groups between n letters.

20. The group $G(a)$ generated by a single substitution a contains the μ different powers of a, μ denoting the order of a (No. 12):

$$a,\ a^2,\ a^3 \ldots a^\mu = 1.$$

These μ powers constitute by themselves a group, since $a^\alpha . a^\beta = a^{\alpha+\beta}$, and this is then the group $G(a)$.

Such a group, which consists of the different powers of a single substitution, is called a *cyclic group.*

As a corollary, we mention that every group contains the substitution 1.

In the case where a is a circular substitution of μ letters, say for instance

$$a = (x_1,\ x_2,\ \ldots\ldots x_\mu),$$

a function which belongs to the cyclic group $G(a)$ is the "cyclic function"

$$(x_1 + \omega x_2 + \omega^2 x_3 + \ldots\ldots \omega^{\mu-1} x_\mu)^\mu,$$

ω being a primitive μ^{th} root of unity.

§4.—*Group and Subgroup.*

21. *Fundamental Theorem:*

If a group G of order N contains all the substitutions of another group H of order P, then N is a multiple of P.

Proof: Let the substitutions of H be denoted by

$$h_1 = 1,\ h_2,\ \ldots\ h_P; \tag{1}$$

the group G contains all the substitutions of H; if it contains no other substitutions, we have $G = H$, $N = 1.P$. If, on the contrary, G still contains another substitution, not contained in H, say g_2, then it must also contain the P products

$$g_2,\ h_2 g_2,\ \ldots\ h_P g_2, \tag{2}$$

according to the definition of a group, and it is easily seen that these substitutions are different from each other as well as from the substitutions (1). Now either the group G contains no other substitutions than the substitutions (1) and (2); then its order N is $2.P$; or G contains still another substitution, say g_3; in which case it must also contain the P products

$$g_3,\ h_2 g_3,\ \ldots\ h_P g_3, \tag{3}$$

which are again different from each other as well as from the substitutions (1) and (2).

Now either G contains no other substitutions; in which case $N = 3.P$ or etc. Continuing in this way, we must finally arrive at a last line

$$g_\nu,\ h_2 g_\nu,\ \ldots\ h_P g_\nu,$$

so that $N = \nu.P$, Q. E. D. (For the details of the proof we refer to Serret, No. 425.)

Besides we have the result:

The $N = \nu.P$ substitutions of G can be arranged in the rectangular table:

$$\begin{array}{llll} h_1 = 1, & h_2, & \ldots & h_P, \\ g_2, & h_2 g_2, & \ldots & h_P g_2, \\ \cdots & \cdots & \cdots & \cdots \\ g_\nu, & h_2 g_\nu, & \ldots & h_P g_\nu. \end{array}$$

22. *Remarks:* 1). The "multipliers" $g_1 = 1,\ g_2,\ \ldots\ g_\nu$ may be chosen in

various ways; the only condition which they must fulfil is that no two of them satisfy a relation of the form $g_\alpha g_\beta^{-1} = h$.*

2). We shall call the integer $\nu = \dfrac{N}{P}$ the *index of the subgroup H under G*.†

3). A similar table can be constructed by multiplying on the left-hand side instead of the right-hand side.

Corollaries: 1). The special case $G = G_{n!}$ gives the corollary:

The order of any substitution-group between n letters is a divisor of n!

2). The special case $H = G(a)$ gives

The order of any substitution contained in a group of order N is a divisor of N.

An important consequence is: If the order N is *prime*, G must itself be a cyclic group consisting of the N different powers of a substitution of the N^{th} order.

This is a special case of the following important theorem due to Cauchy:

If the order N of a group G is divisible by a prime number p, then G contains a cyclic subgroup of order p.

For the proof, which is pretty complicated, we refer to Jordan, No. 40–42; a further extension has been given by Sylow (see Mathematische Annalen, Bd. 5, besides Netto, §48, and Frobenius, Borchardt's Journal, Bd. 100).

23. Let now ψ be a rational function of $x_1, x_2, \ldots . x_n$ which belongs to the subgroup H, and let us apply to it all the N substitutions of G; then we have $\psi_{hg_\beta} = \psi_{g_\beta}$, since $\psi_h = \psi$, consequently the P substitutions of the line

$$g_\beta,\ h_2 g_\beta,\ \ldots .\ h_P g_\beta$$

of our table change the function ψ into the same new function $\psi_\beta = \psi_{g_\beta}$. Further, it is easily seen, from the properties of the multipliers g, that two functions ψ_{g_β} and ψ_{g_γ} are different if β is different from γ. Hence

A rational function ψ which belongs to the subgroup H takes $\nu = \dfrac{N}{P}$ different values, if operated upon by all the N substitutions of the group G.

These ν values

$$\psi_1 = \psi,\ \psi_2 = \psi_{g_2},\ \ldots .\ \psi_\nu = \psi_{g_\nu}$$

* h stands here and afterwards for "any substitution of the group H."

† Serret uses the expression index of a group for what we should call its index under the symmetric group.

are called the ν *conjugate values of* ψ *under the group* G ("gleichberechtigt," Klein).

The special case $G = G_{n!}$ gives the corollary:

The number ρ *of different values which a rational function assumes if operated upon by all the* $n!$ *substitutions, is always a divisor of* $n!$:

$$\rho = \frac{n!}{P}. \quad \text{(Lagrange, compare No. 5.)}$$

Example: $n = 4$, notations of No. 15,

$$G = G_{12}, \quad H = G_4, \quad \psi = (x_1 - x_2)(x_3 - x_4),$$

$$\begin{array}{llll|l} 1; & a = (12)(34); & b = (13)(24); & c = (14)(23) & \psi_1 \\ d = (234); & ad = (132); & bd = (143); & cd = (124) & \psi_2 \\ d^2 = (243); & ad^2 = (142); & bd^2 = (123); & cd^2 = (134) & \psi_3 \end{array}$$

$$\psi_1 = (x_1 - x_2)(x_3 - x_4); \quad \psi_2 = (x_1 - x_3)(x_4 - x_2); \quad \psi_3 = (x_1 - x_4)(x_2 - x_3).$$

Other examples have already occurred in No. 8 and 9.

24. *Excursus on Transitive Groups.*

The distinction between transitive and intransitive groups may be easiest understood by some examples.

Let us first consider the group G_8 between four letters of No. 15, and fix our attention upon one of the four letters, say x_1; then there exist in G_8 substitutions which replace x_1 by x_1 (for instance 1), substitutions which replace x_1 by x_2 (for instance (x_1x_2)), by x_3 (for instance $(x_1x_3)(x_2x_4)$), by x_4 (for instance $(x_1x_4)(x_2x_3)$). Hence the substitutions of G_8 allow to replace x_1 by any one of the four letters x_1, x_2, x_3, x_4; such a group is called a *transitive group.*

On the other hand, the group H_4 of No. 15 contains substitutions which replace x_1 by x_1 or by x_2, but no substitution which replaces x_1 by x_3 or by x_4; such a group is called *intransitive.*

The definition in the case of n letters is analogous (Netto, §61).

The order of a transitive group between n *letters is always divisible by* n.

Proof: Let us select, among the substitutions of the given group G, all those which leave one of the n letters, say x_1, unaltered; evidently these substitutions constitute by themselves a group H, subgroup of G. Let us now apply the theorem of No. 21 to the two groups G and H, and construct the rectangular table of No. 21. The substitution g_2 replaces x_1 by some other letter, say

x_2; then all the substitutions of the second line, and no other substitutions of G, will replace x_1 by x_2, and similarly for the other lines. Hence, if moreover G is transitive, ν must be equal to n, which proves our proposition (compare Netto, §62).

Examples: $n = 3$: G_3, G_6 are transitive,

$n = 4$: G_{24}, G_{12}, G_8, G_4 are transitive.

The lowest possible order of a transitive group between n letters is therefore n; a transitive group between n letters of order n is called a *regular group*. Regular groups are for instance $n = 3$, G_3; $n = 4$, G_4, and the cyclic group: 1; (1234); (13)(24); (1432).

§5.—*Algebraic Relations between Different Rational Functions of the Roots.*

25. The series of theorems on algebraic relations between different rational functions of $x_1, x_2 \ldots . x_n$ begins with the well-known theorem on symmetric functions:

Every rational symmetric function of $x_1, x_2, \ldots . x_n$ is rationally expressible in terms of the elementary symmetric functions

$$c_1 = x_1 + x_2 + \ldots . + x_n,$$
$$c_2 = x_1x_2 + x_1x_3 + \ldots . + x_{n-1}x_n,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$c_n = x_1x_2 \ldots . x_n,$$
$$\text{Symm.}(x_1, x_2, \ldots . x_n) = \text{Rat.}(c_1, c_2, \ldots . c_n).$$

If, in particular, the symmetric function is an *integral function* of $x_1, x_2 \ldots x_n$ *with integral coefficients*, then also its expression in terms of $c_1, c_2, \ldots . c_n$ is an integral function with integral coefficients (see Serret, No. 174).

26. *Theorem:*

If an asymmetric function $\phi(x_1, x_2, \ldots . x_n)$ takes ρ different values, $\phi_1 = \phi, \phi_2, \ldots . \phi_\rho$ when operated upon by all the $n!$ substitutions, then ϕ satisfies an equation of the ρ^{th} degree whose coefficients are rational functions of $c_1, c_2, \ldots . c_n$.

For if the ρ values of ϕ,

$$\phi_1, \phi_2, \ldots . \phi_\rho$$

are simultaneously operated upon by the same substitution, they are only interchanged among themselves (compare No. 18 and below No. 38); hence any symmetric function of $\phi_1, \phi_2, \ldots . \phi_\rho$ remains unaltered by all the $n!$ substitu-

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tions and is therefore a symmetric function also of $x_1, x_2, \ldots. x_n$ and consequently rationally expressible in terms of $c_1, c_2, \ldots. c_n$.

Hence the coefficients of the equation

$$(\phi - \phi_1)(\phi - \phi_2) \ldots. (\phi - \phi_\rho) = 0,$$

whose roots are the ρ values of ϕ, are rationally expressible in terms of $c_1, c_2, \ldots. c_n$. Q. E. D.

Such an equation is called a *resolvent equation*, or simply resolvent of the equation

$$x^n - c_1 x^{n-1} + c_2 x^{n-2} \ldots. + c_n = 0,$$

because in certain cases the solution of the resolvent implies the solution of the given equation.

Examples have already occurred in Nos. 2 and 4.

27. Let us next consider a rational function $\phi(x_1, x_2, \ldots. x_n)$ belonging to a group G, and another function ψ belonging to a subgroup H of G of index ν.

We shall always indicate these frequently recurring assumptions by the diagram

$$\begin{array}{c} G;\ \phi \\ \nu \Big| \\ H;\ \psi. \end{array}$$

If, then, both functions are operated upon by all the substitutions of G, ϕ remains unaltered, whereas ψ takes ν conjugate values (No. 23).

We are going to prove that, accordingly, ϕ is rationally expressible in terms of ψ (and of $c_1, c_2, \ldots. c_n$), whereas the ν conjugate values of ψ satisfy a "resolvent equation" of the ν^{th} degree whose coefficients are rational functions of ϕ (and $c_1, c_2, \ldots. c_n$).

a). In order to prove the first part of this proposition, let us form the product $\phi\psi^m$, m being any integer; this function will then, according to our assumptions, remain unaltered by all the substitutions of H and—leaving exceptional cases aside—by no other substitutions. Hence, denoting by ρ the index of the group H under the symmetric group $G_{n!}$, the function $\phi\psi^m$ takes ρ different values if operated upon by all the $n!$ substitutions; let them be denoted by

$$\phi_1\psi_1^m = \phi\psi^m,\ \phi_2\psi_2^m,\ \ldots.\ \phi_\rho\psi_\rho^m,$$

ϕ_α and ψ_α being derived from ϕ and ψ by the same substitution;* the sum

$$\phi_1\psi_1^m + \phi_2\psi_2^m + \ldots. \phi_\rho\psi_\rho^m = s_m \tag{1}$$

* In general, some of the ϕ_α's will be equal, but this does not affect the proof.

is then a symmetric function of $x_1, x_2, \ldots. x_n$, and is therefore rationally expressible in terms of $c_1, c_2, \ldots. c_n$.

b). Forming now the equation (1) successively for $m = 0, 1, 2 \ldots. \rho - 1$, we obtain the following system of ρ equations linear with respect to $\phi_1, \phi_2, \ldots \phi_\rho$:

$$\left.\begin{array}{l} \phi_1 + \phi_2 \quad + \ldots. \quad \phi_\rho \quad = s_0, \\ \psi_1\phi_1 + \psi_2\phi_2 + \ldots. \psi_\rho\phi_\rho = s_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \psi_1^{\rho-1}\phi_1 + \psi_2^{\rho-1}\phi_2 + \ldots. \psi_\rho^{\rho-1}\phi_\rho = s_{\rho-1} \end{array}\right\} \tag{2}$$

Solving these equations with respect to ϕ_1, we obtain ϕ_1 expressed as a rational function of $\psi_1, \psi_2, \ldots. \psi_\rho$; $c_1, c_2, \ldots. c_n$, symmetric with respect to the $\rho - 1$ quantities $\psi_2, \psi_3, \ldots. \psi_\rho$. But any symmetric function of these quantities can be transformed into a function of ψ, and symmetric functions of $\psi_1, \psi_2, \ldots. \psi_\rho$ and consequently also of $\psi_1, c_1, c_2, \ldots. c_n$. Q. E. D.

For the details of the proof we refer to Serret, No. 492, and Netto, §§95, 96.

This fundamental proposition is due to Lagrange (Reflexions sur la resolution algebrique des équations, §100, Œuvres, Vol. III) and called *Lagrange's Theorem;* it is usually expressed in the form

If a rational function $\phi(x_1, x_2, \ldots. x_n)$ *remains unaltered by all the substitutions which leave another rational function* $\psi(x_1, x_2, \ldots. x_n)$ *unaltered, then* ϕ *is rationally expressible in terms of* ψ.

$$\phi = \text{Rat.}\,(\psi; c_1, c_2, \ldots. c_n).$$

28. *Remarks:* The expression of ϕ in terms of ψ is found to have the form

$$\phi = \frac{g(\psi; c_1, c_2, \ldots. c_n)}{\Delta_\psi}.$$

The *denominator* Δ_ψ denotes the square of the product of all the differences of the ρ values $\psi_1, \psi_2, \ldots. \psi_\rho$:

$$\Delta_\psi = (\psi_1 - \psi_2)^2(\psi_1 - \psi_3)^2 \ldots. (\psi_{\rho-1} - \psi_\rho)^2.$$

It is a symmetric function of $x_1, x_2, \ldots. x_n$, hence rationally expressible in terms of $c_1, c_2, \ldots. c_n$ and called the *discriminant of the function* ψ. It is different from zero excepting for those special values of the x's for which two or more of the functions $\psi_1, \psi_2, \ldots. \psi_\rho$ become equal (see below, No. 67).

The *numerator* $g(\psi; c_1 \ldots. c_n)$ is an integral function of $\psi, c_1, c_2, \ldots. c_n$ with rational coefficients, if ϕ and ψ themselves are *integral* functions with *rational coefficients* of $x_1, x_2, \ldots. x_n$.

The special case $G = H$ furnishes the *Corollary* I:

All rational functions that belong to the same group are rationally expressible in terms of any one of them.

Further, the special case $H = 1$ furnishes the *Corollary* II:

Every rational function of $x_1, x_2, \ldots . x_n$ is rationally expressible in terms of any $n!$-valued function, such as for instance (No. 7):

$$V = m_1 x_1 + m_2 x_2 + \ldots . m_n x_n .$$

29. The second part of the proposition announced in the beginning of No. 27 is now an immediate consequence of the first. For it is easily seen that the ν conjugate values of ψ under G,

$$\psi_1 = \psi_1, \ \psi_2, \ldots . \psi_\nu$$

are only interchanged among each other if simultaneously operated upon by any substitution of G (see also below, No. 38). Hence the coefficients of the equation

$$(\psi - \psi_1)(\psi - \psi_2) \ldots . (\psi - \psi_\nu) = 0,$$

being symmetric functions of $\psi_1, \psi_2, \ldots . \psi_\nu$ remain unaltered by all the substitutions of the group G of the function ϕ, and are therefore rationally expressible in terms of ϕ, according to Lagrange's theorem. Thus we have the result:

The ν conjugate values of ψ under G satisfy a resolvent equation of the ν^{th} degree whose coefficients are rational functions of ϕ and of $c_1, c_2, \ldots . c_n$:

$$\psi^\nu - R_1(\phi;\ c_1 \ldots . c_n)\psi^{\nu-1} + \ldots . \pm R_\nu(\phi;\ c_1 \ldots . c_n) = 0 .$$

§6.—*Solution of an Equation by a Chain of Resolvent Equations.*

30. Let us now reconsider, in the light of these theorems, the solution of the cubic equation

$$x^3 + px + q = 0 . \tag{1}$$

Using the notations of No. 1, and remembering the relation (3) of No. 1, we may split up Cardan's formula into the "chain of binomial equations":

$$\xi^2 = \frac{q^2}{4} + \frac{p^3}{27}, \tag{2}$$

$$\eta^3 = -\frac{q}{2} + \xi, \tag{3}$$

$$x_1 = \eta - \frac{p}{3\eta}; \ x_2 = \omega\eta - \frac{\omega^2 p}{3\eta}; \ x_3 = \omega^2\eta - \frac{\omega p}{3\eta}, \tag{4}$$

and we have (No. 1):

$$\left.\begin{aligned} \xi &= \frac{\sqrt{-3}}{18}(x_2 - x_3)(x_3 - x_1)(x_1 - x_2), \\ \eta &= \frac{x_1 + \omega^2 x_2 + \omega x_3}{3}. \end{aligned}\right\} \tag{5}$$

The solution may therefore be described as follows: *Given* are the coefficients, belonging to the group G_6 (notation of No. 14); by solving a resolvent of the second degree (2), we find the two-valued function ξ, which belongs to the subgroup G_3 of G_6 (No. 26); hence, by solving a resolvent of the third degree (3), which in this case turns out to be binomial, we obtain the six-valued function η, which belongs to the subgroup G_1 of G_3 (No. 29); finally, the roots x_1, x_2, x_3, which belong to the groups G_2', G_2'', G_2''' respectively, are rationally expressible in terms of η according to No. 27.

The solution may therefore be exhibited by the diagrams (No. 27):

$$\begin{array}{c} G_6;\ p,\ q \\ 2\ \big| \\ G_3;\ \xi \\ 3\ \big| \\ G_1;\ \eta \end{array} \qquad\qquad \begin{tikzcd}[column sep=small] & G_2'';\ x_2 \arrow[d, dash] & \\ G_2';\ x_1 \arrow[dr, dash] & & G_2''';\ x_3 \arrow[dl, dash] \\ & G_1;\ \eta & \end{tikzcd}$$

31. This solution of the cubic equation suggests now a general plan* for the solution of the equation of the n^{th} degree:

$$x^n - c_1 x^{n-1} + c_2 x^{n-2} \ldots \pm c_n = 0. \tag{1}$$

The *complete* solution of this equation requires to find not only one or some of its roots, but *all its n roots* $x_1, x_2, \ldots x_n$. If, therefore, we agree to consider a quantity *as known* when it can be derived from known quantities by rational operations, *the solution of* (1) *is equivalent to the determination of one $n!$-valued function* (No. 7):

$$V = m_1 x_1 + m_2 x_2 \ldots m_n x_n.$$

For as soon as the n roots are known, V is known, and conversely, since, according

* In fact the only possible one, as we shall see later on, when we give the complete theory of the solution by radicals (No. 85); here we use the problem only as a guide in our research on substitution-groups.

to Lagrange's theorem (No. 28, II), the roots are rationally expressible in terms of V:

$$x_a = \text{Rat.}\,(V;\ c_1,\ c_2,\ \ldots\ldots c_n). \tag{2}$$

Suppose now we could interpolate between the symmetric group $G_{n!}$ of the coefficients and the group 1 of V a series of groups, $H, I, \ldots\ldots M$, such that each group is a subgroup of the group immediately preceding it; let λ be the index of H under G, μ that of I under H, and so forth, finally ρ that of 1 under M, and let $\xi, \eta, \ldots \psi$ be rational functions belonging to the groups $H, I, \ldots M$ respectively (No. 18), as represented in the adjoined diagram.

$$\left.\begin{array}{l} G_{n!};\ c_1,\ c_2,\ \ldots\ldots c_n \\ \lambda \Big| \\ H\ ;\ \xi \\ \mu \Big| \\ I\ ;\ \eta \\ \cdot \\ \cdot \\ \cdot \\ M\ ;\ \psi \\ \rho \Big| \\ 1\ ;\ V \end{array}\right\} \tag{3}$$

Applying, then, to the functions $\xi, \eta, \ldots\ldots \psi, V$ successively the theorems of No. 26 and No. 29 we obtain a "*chain of resolvent equations*"

$$\left.\begin{array}{l} \xi^\lambda - A_1(\quad c_1,\ c_2,\ \ldots\ldots c_n)\,\xi^{\lambda-1} + \ldots\ldots = 0, \\ \eta^\mu - B_1(\xi;\ c_1,\ c_2,\ \ldots\ldots c_n)\,\eta^{\mu-1} + \ldots\ldots = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ V^\rho - E_1(\psi;\ c_1,\ c_2,\ \ldots\ldots c_n)\,V^{\rho-1} + \ldots\ldots = 0. \end{array}\right\} \tag{4}$$

The reduction of the given equation (1) to this chain of equations will be a real simplification:

1). If the indices $\lambda, \mu, \ldots\ldots \rho$ are all of them *smaller than* n.

2). If the equations of our chain should turn out to be *binomial.*

The first of these cases can always be reduced to the second, as we shall see (below, No. 86, Remark).

32. In the case $n = 4$ it is easy to select, among the groups enumerated in No. 15–17, in various ways, a series of intermediate groups between G_{24} and

$G=1$ such that all the indices are smaller than four; for instance, using the notations of No. 15–17:

$$\begin{array}{l} G_{24};\ c_1,\ c_2,\ c_3,\ c_4 \\ 2\,\Big| \\ G_{12};\ \phi=\Pi\,(x_\alpha-x_\beta)=\sqrt{\Delta}, \\ 3\,\Big| \\ G_4\ ;\ \psi=(x_1-x_2)(x_3-x_4), \\ 2\,\Big| \\ G_2\ ;\ \chi=[m_1(x_1-x_2)+m_3(x_3-x_4)]^2, \\ 2\,\Big| \\ G_1\ ;\ V=m_1(x_1-x_2)+m_3(x_3-x_4), \end{array}$$

where $G_2=[1;\,(1\,2)(3\,4)]$.

Or we may descend from G_{24} to the group G_8 with the function

$$\xi=x_1x_2+x_3x_4;$$

hence to the group H_4 with the function

$$\eta=x_1+x_2-x_3-x_4;$$

hence to G_2 and G_1 as above.

Also Ferrari's solution, though apparently proceeding in a different way, will later on be seen to repose essentially on the same principle (see below, No. 87, where the definitive solution will be given).

In the case of the cubic equation, it would not have been possible to choose the intermediate groups such that the indices are all smaller than three; here the solution is founded on the existence of a binomial resolvent between G_1 and G_3; generally the question arises—*under what conditions will all the resolvent equations of the chain* (4) *in No.* 31 *become binomial equations*, and, as we may add without loss of generality, binomial equations *of prime degrees*, since a binomial equation of composite degree can always be resolved into a chain of binomial equations of prime degrees? (Answer in No. 37, 42.)

§7.—*Conjugate Groups.*

33. Each of the ν conjugate functions $\psi_1, \psi_2, \ldots . \psi_\nu$ of No. 23 belongs to a certain group; in order to determine them, let us first determine generally the group of the function $\psi' = \psi_s$ into which ψ is changed by any substitution whatever, s.

Let h' be any substitution which leaves ψ' unaltered, so that

$$\psi'_{h'} = \psi', \text{ that is } \psi_{sh'} = \psi_s;$$

operated upon by s^{-1} this identity becomes $\psi_{sh's^{-1}} = \psi$, consequently $sh's^{-1}$ leaves ψ unaltered and belongs therefore to the group H of ψ:

$$sh's^{-1} = h; \text{ hence } h' = s^{-1}hs.$$

Conversely, any substitution of the form $s^{-1}hs$ leaves ψ_s unaltered. Hence

If ψ belongs to the group

$$H = [h_1 = 1, h_2, \ldots . h_P],$$

then ψ_s belongs to the group

$$[s^{-1}h_1s = 1, s^{-1}h_2s, \ldots . s^{-1}h_Ps],$$

which is denoted by $s^{-1}Hs$.

The substitution $s^{-1}hs$ is called *the transformed of the substitution* h by the substitution s; the group $s^{-1}Hs$ *the transformed of the group* H by the substitution s (Netto, §§45, 46).

34. There exists a simple rule to derive the transformed $s^{-1}hs$ from h without actually effecting the multiplication. Suppose first h to be a circular substitution, for instance

$$h = (\alpha\, \beta\, \gamma\, \delta),$$

whereas s is any substitution whatever

$$s = \begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & \ldots . & \lambda \\ \alpha' & \beta' & \gamma' & \delta' & \varepsilon' & \ldots . & \lambda' \end{pmatrix},$$

$\alpha\beta\gamma\delta\varepsilon \ldots . \lambda$ and $\alpha'\beta'\gamma'\delta'\varepsilon' \ldots . \lambda'$ being any two permutations of the numbers $123 \ldots . n$. Then

$$s^{-1} = \begin{pmatrix} \alpha' & \beta' & \gamma' & \delta' & \varepsilon' & \ldots . & \lambda' \\ \alpha & \beta & \gamma & \delta & \varepsilon & \ldots . & \lambda \end{pmatrix},$$

hence

$$s^{-1}hs = \begin{pmatrix} \alpha' & \beta' & \gamma' & \delta' & \varepsilon' & \ldots . & \lambda' \\ \beta' & \gamma' & \delta' & \alpha' & \varepsilon' & \ldots . & \lambda' \end{pmatrix} = (\alpha'\, \beta'\, \gamma'\, \delta').$$

To find the transformed of any other substitution, we decompose it into its circular factors and notice that

$$s^{-1}hh's = s^{-1}hs \cdot s^{-1}h's,$$

then we obtain the rule: *The transformed $s^{-1}hs$ can be derived from h by operating the substitution s within the cycles of h.*

Example: $h \quad = (1\,2\,3)(4\,5)$; $s = (3\,4)(2\,5)$,
$s^{-1}hs = (1\,5\,4)(3\,2)$.

35. Returning now to the notations and assumptions of No. 23, and applying the lemma of No. 33 to the ν functions which are conjugate with ψ under G, we have the result:

The ν conjugate values of ψ under G, viz.

$$\psi_1 = \psi,\ \psi_2 = \psi_{g_2}, \ldots \psi_\nu = \psi_{g_\nu} \tag{1}$$

belong to the groups

$$H_1 = H,\ H_2 = g_2^{-1}Hg_2 \ldots H_\nu = g_\nu^{-1}Hg_\nu \tag{2}$$

respectively.

All these groups are subgroups of G; they are said to be *conjugate subgroups* of G ("gleichberechtigte Untergruppen," Klein).

If g denote any substitution of G, then ψ_g must be equal to one of the ν conjugate functions (1) (No. 23), viz. $\psi_g = \psi_\alpha$, if $g = hg_\alpha$; hence the group of ψ_g must be identical with the group of ψ_α, that is

$$g^{-1}Hg = g_\alpha^{-1}Hg_\alpha; \tag{3}$$

the group $g^{-1}Hg$ is therefore likewise conjugate with H under G.

Similarly two substitutions a and a' of G are called *conjugate substitutions* of G ("gleichberechtigte Substitutionen," Klein), if the one is the transformed of the other *by a substitution of G*:

$$a' = g^{-1}ag. \tag{4}$$

For instance, the two substitutions $a = (2\,3\,4)$ and $a' = (1\,4\,3)$ are conjugate under the alternate group of four letters G_{12}, since $(1\,4\,3)$ is the transformed of $(2\,3\,4)$ by the substitution $(1\,2)(3\,4)$ *which belongs to* G_{12}; $(2\,3\,4)$ and $(1\,3\,4)$ on the contrary are not conjugate under G_{12}, but they are conjugate under the symmetric group G_{24}, since

$$(1\,3\,4) = (1\,2)^{-1}(2\,3\,4)(1\,2).$$

36. In general the ν conjugate subgroups $H_1, H_2, \ldots, H_\nu$ are distinct; but it may happen that they all coincide:

$$H = g_2^{-1}Hg_2 = \ldots = g_\nu^{-1}Hg_\nu, \tag{5}$$

so that the ν conjugate functions (1) belong all to the same group H. In this case the group H is called a *self-conjugate subgroup* of G.*

From (3) it follows that *the transformed of a self-conjugate subgroup H of G by any substitution g of G is identical with H*:

$$g^{-1}Hg = H. \tag{6}$$

It may be well to remark that this equation stands for P equations of the form

$$g^{-1}h_\alpha g = h_{i_\alpha}, \qquad (\alpha = 1, 2 \ldots P) \tag{7}$$

$i_1, i_2 \ldots i_P$ being some permutation of the numbers $1, 2 \ldots P$. Hence *if a self-conjugate subgroup H of G contain a substitution h, it also contains all the substitutions which are conjugate with h under G.*

Examples of conjugate and self-conjugate subgroups: 1). $n = 3$ (notations of No. 14):

a). $G = G_6$, $H = G_3$; multipliers: 1, $g_2 = (23)$,
$\psi_1 = \psi = (x_1 + \omega x_2 + \omega^2 x_3)^3$; $\psi_2 = \psi_{g_2} = (x_1 + \omega^2 x_2 + \omega x_3)^3$,
$H_1 = [1;\ (123);\ (132)] = G_3$;
$H_2 = [1;\ (132);\ (123)] = G_3$.

Hence G_3 is a self-conjugate subgroup of G_6 and ψ_1 and ψ_2 belong to the same group.

b). $G = G_6$; $H = G_2'$; multipliers: 1, $g_2 = (12)$, $g_3 = (13)$,
$H_1 = \quad [1, (23)] = G_2'$,
$H_2 = g_2\,[1, (13)] = G_2''$,
$H_3 = g_3\,[1, (21)] = G_2'''$.

Hence G_2' is not self-conjugate under G_6.

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*I adopt here Cole's translation of "ausgezeichnete Untergruppe," Klein (see On Klein's Ikosaeder, Amer. Journal, Vol. IX); Jordan's expression is: H is "permutable à toutes les substitutions de G."

2). $n = 4$ (notations of No. 15–17):

$G = G_{12}$, $H = G_4$; multipliers: 1, $g_2 = (2\,3\,4)$, $g_3 = (2\,4\,3)$,
$\psi_1 = (x_1 - x_2)(x_3 - x_4)$; $\psi_2 = (x_1 - x_3)(x_4 - x_2)$; $\psi_3 = (x_1 - x_4)(x_2 - x_3)$,
$H_1 = [1;\ (1\,2)(3\,4);\ (1\,3)(2\,4);\ (1\,4)(2\,3) = G_4$,
$H_2 = [1;\ (1\,3)(4\,2);\ (1\,4)(3\,2);\ (1\,2)(3\,4) = G_4$,
$H_3 = [1;\ (1\,4)(2\,3);\ (1\,2)(4\,3);\ (1\,3)(4\,2) = G_4$.

Hence G_4 is a self-conjugate subgroup of G_{12}.

G_8 on the other hand is not self-conjugate under G_{24}, since the three conjugate subgroups G_8, G_8', G_8'' are distinct (see No. 15).

37. Let us now return to the question proposed at the end of No. 32 and ask: *Under what conditions will the resolvent equation of the ν^{th} degree,*

$$g(\psi;\ \phi) = 0, \tag{1}$$

which, according to No. 29, is satisfied by the ν conjugate values of ψ under G:

$$\psi_1 = \psi,\ \psi_2 \ldots \psi_\nu \tag{2}$$

turn out to be a binomial equation?

Suppose it were binomial:

$$\psi^\nu = R(\phi), \tag{3}$$

then its roots (2) may be written

$$\psi_1 = \psi,\ \psi_2 = \omega\psi,\ \psi_3 = \omega^2\psi,\ \ldots \psi_\nu = \omega^{\nu-1}\psi, \tag{4}$$

ω being a primitive ν^{th} root of unity and the order of the conjugate values being chosen conveniently.

Hence the ν conjugate functions (2) differ only by constant factors from each other and must therefore all belong to the same group; consequently their groups, viz. the ν conjugate groups $H_1, H_2, \ldots H_\nu$ must coincide; that is, *the group H of the function ψ must be a self-conjugate subgroup of G* (No. 36).

Hence follows that in the solution of the cubic (biquadratic) equation it is impossible to descend directly from the group G_6 (G_{24}) to the subgroup G_2 (G_8) by a binomial resolvent (see No. 36).

We ask next: *Is this condition for a binomial resolvent also sufficient?* (Answer below, No. 42.)*

*References concerning §8: Netto, §45, 46, 77, 103.

§8.—*Substitution-Group between the ν Conjugate Values $\psi_1, \psi_2, \ldots \psi_\nu$.*

38. Retaining still the assumptions and notations of No. 23, let us apply simultaneously to the ν conjugate functions

$$\psi_1 = \psi, \ \psi_2 = \psi_{g_2} \ldots \psi_\nu = \psi_{g_\nu} \tag{1}$$

any substitution g of G; they are changed into

$$\psi_g, \psi_{g_2 g} \ldots \psi_{g_\nu g}. \tag{2}$$

These values are a permutation of the values (1), for since $g_a g$ belongs again to the group G, $\psi_{g_a g}$ must be one of the values (1); and besides, the values (2) are all distinct since the identity $\psi_{g_a g} = \psi_{g_\beta g}$ would imply $g_a g_\beta^{-1} = h$, against No. 22. Thus to every substitution g of G between the letters $x_1, x_2, \ldots x_n$ corresponds one definite substitution γ between the letters $\psi_1, \psi_2, \ldots \psi_\nu$, viz.

$$\gamma = \begin{pmatrix} \psi & \psi_{g_2} & \ldots \psi_{g_\nu} \\ \psi_g & \psi_{g_2 g} & \ldots \psi_{g_\nu g} \end{pmatrix},$$

which we shall for shortness write

$$\gamma = \begin{pmatrix} \psi_{g_a} \\ \psi_{g_a g} \end{pmatrix}.$$

On the whole we obtain, then, a system Γ of N substitutions γ, among which, however, some may be equal (see below, No. 39).

This system Γ of substitutions forms a group.

For let g' be another substitution of G, the corresponding substitution γ' will be

$$\gamma' = \begin{pmatrix} \psi_{g_a} \\ \psi_{g_a g'} \end{pmatrix}$$

and the substitution γ'' which corresponds to the product gg' is

$$\gamma'' = \begin{pmatrix} \psi_{g_a} \\ \psi_{g_a g g'} \end{pmatrix};$$

but this is equal to the product $\gamma\gamma'$ since we may write γ' in the form

$$\gamma' = \begin{pmatrix} \psi_{g_a g} \\ \psi_{g_a g g'} \end{pmatrix}.$$

by conveniently changing the order of the letters in the first and accordingly in the second line (see No. 6), hence

$$\gamma\gamma' = \begin{pmatrix} \psi_{g_a} \\ \psi_{g_a g g'} \end{pmatrix} = \gamma''.$$

Let g be any substitution of G, not contained in H, then the corresponding substitution of Γ is some power of γ, say $\gamma^\varkappa$, different from 1, that is, $\varkappa$ is not divisible by ν. On account of the isomorphism between G and Γ (No. 38), to the power g^ν corresponds then the power $(\gamma^\varkappa)^\nu$, which is $= 1$, since $\gamma^\nu = 1$; hence follows that g^ν belongs to the subgroup H (see No. 39, *b*).

Conversely—suppose some power g^m belongs to H, then the corresponding substitution $(\gamma^\varkappa)^m$ must be equal to 1, therefore $m \equiv 0 \pmod{\nu}$, since $\varkappa$ is not divisible by ν.

Thus we have the result:

If H be a self-conjugate subgroup of G of prime index ν, and g any substitution of G not contained in H, then g^ν, and no lower power of g, belongs to H.

Further, *the order μ of g must be a multiple of ν*; for if we reduce μ to the form

$$\mu = q\nu + \mu' \qquad (0 \overline{\gtrless} \mu' < \nu),$$

then we have, since $g^\nu = h$,

$$g^\mu = h^q g^{\mu'} = 1;$$

hence $g^{\mu'}$ belongs to H, therefore $\mu' = 0$, since it is $< \nu$, according to the above theorem.

41. *Examples to No.* 39:

1). $n = 3$ (notations of Nos. 8 and 14):

$$G = G_6,\ H = G_1,\ I = G_1,\ \psi = m_1 x_1 + m_2 x_2 + m_3 x_3.$$

Denoting

$$\psi_1 = \psi,\ \psi_2 = \psi_a,\ \psi_3 = \psi_b,\ \psi_4 = \psi_c,\ \psi_5 = \psi_d,\ \psi_6 = \psi_e,$$

we have the following holoedric isomorphism between the two groups G and Γ:

G	Γ
1	1
$(x_1 x_2 x_3)$	$(\psi_1 \psi_2 \psi_3)(\psi_4 \psi_6 \psi_5)$
$(x_1 x_3 x_2)$	$(\psi_1 \psi_3 \psi_2)(\psi_4 \psi_5 \psi_6)$
$(x_2 x_3)$	$(\psi_1 \psi_4)(\psi_2 \psi_5)(\psi_3 \psi_6)$
$(x_1 x_3)$	$(\psi_1 \psi_5)(\psi_2 \psi_6)(\psi_3 \psi_4)$
$(x_1 x_2)$	$(\psi_1 \psi_6)(\psi_2 \psi_4)(\psi_3 \psi_5)$

2). $n = 4$ (notations of Nos. 15, 16).

$G = G_{12},\ H = G_4,\ I = G_4,$

$\psi_1 = (x_1 - x_2)(x_3 - x_4)$; $\psi_2 = (x_1 - x_3)(x_4 - x_2)$; $\psi_3 = (x_1 - x_4)(x_2 - x_3)$.

Meriedric isomorphism:

G				Γ
1	; (12)(34);	(13)(24);	(14)(23)	1
(234);	(132)	; (143)	; (124)	$(\psi_1\psi_2\psi_3)$
(243);	(142)	; (123)	; (134)	$(\psi_1\psi_3\psi_2)$

3). $n=4$; $G=G_{24}$, $H=G_8$, $I=G_4$,

$$\psi_1=x_1x_2+x_3x_4;\ \psi_2=x_1x_3+x_2x_4;\ \psi_3=x_1x_4+x_2x_3.$$

The order of Γ is $\frac{N}{Q}=\frac{24}{4}=6$; Γ is the symmetric group between the three letters ψ_1, ψ_2, ψ_3.

42. Let us now take up the converse of the theorem of No. 37 concerning binomial resolvent equations, confining ourselves, however, to the case where *the index ν is prime*, the only case of importance for our purposes. We shall prove:

If H be a self-conjugate subgroup of G of prime index ν, then it is always possible to construct a rational function χ for which the resolvent equation $g(\psi, \phi)=0$ of No. 29 becomes binomial:

$$\chi^\nu=R(\phi).$$

For let ψ be *any* function belonging to the group H, and $\psi_1, \psi_2 \ldots . \psi_\nu$ its ν conjugate values under G; if, then, the letters $x_1, x_2, \ldots . x_n$ are operated upon by all the substitutions of G, the functions $\psi_1, \psi_2, \ldots . \psi_\nu$ undergo a group Γ of substitutions which is, according to No. 39, *c*), the cyclic group

$$\Gamma=[1, \gamma, \gamma^2, \ldots . \gamma^{\nu-1}],$$

where $$\gamma=(\psi_1, \psi_2, \ldots . \psi_\nu),$$

the notation of the ψ's being suitably chosen.

Hence the function

$$(\psi_1+\omega\psi_2+\omega^2\psi_3+\ldots . \omega^{\nu-1}\psi_\nu)^\nu,$$

ω being a primitive ν^{th} root of unity, remains unaltered by the group Γ and consequently also by the group G, if considered as a function of the x's, hence it is expressible rationally in terms of ϕ; $c_1, c_2, \ldots . c_n$ (No. 27).

The expression

$$\chi = \psi_1 + \omega\psi_2 + \omega^2\psi_3 + \ldots\ldots \omega^{\nu-1}\psi_\nu$$

is then a function of $x_1, x_2, \ldots\ldots x_n$ which belongs* to the group H and satisfies a binomial resolvent equation

$$\chi^\nu = R(\phi; c_1, c_2, \ldots\ldots c_n). \qquad \text{Q. E. D.}\dagger$$

§9.—*Decomposition of Groups, in particular of the Symmetric Group.*

43. Every group has two self-evident self-conjugate subgroups, viz. itself and the group 1; if it contain no other self-conjugate subgroup, it is called a *simple group;* otherwise, a *composite group.*

A composite group G can be *decomposed* in the following way: From the group G we descend to a *maximal self-conjugate subgroup* H; that is, a self-conjugate subgroup of G which is not contained in a larger self-conjugate subgroup of G; from H we descend to a maximal self-conjugate subgroup I of H, and so on until at last we arrive at the group 1. Such a series of groups

$$\begin{array}{l} G,\ H,\ I \ldots\ldots M,\ 1, \\ \ \lambda \quad\ \mu \qquad\quad\ \rho \end{array}$$

in which each group is a maximal self-conjugate subgroup of the group immediately preceding it, is called a *series of composition* of the group G, and the successive indices λ (index of H under G), μ (index of I under H) .... ρ are called the *factors of composition.*

Examples: ‡ (Compare No. 36).

1). Symmetric group of three letters:

$$G_6,\ G_3,\ 1.$$

2). Symmetric group of four letters:

$$G_{24},\ G_{12},\ G_4,\ G_2,\ 1;$$

instead of the group

$$G_2 = [1,\ (1\,2)(3\,4)],$$

---

* ψ can always be chosen such (viz. by replacing, if necessary, ψ by $\psi\ (m + \psi)$) that χ remains unaltered by *no other* substitutions than those of G (see also Netto, §102).

† References concerning §9: Jordan, No. 67–74; Netto, §87–93; König, l. c. §1; Capelli, l. c. III.

‡ From No. 36 it follows easily that a subgroup of index 2 is always a self-conjugate subgroup.

we may also choose the similar groups

$$G_2' = [1, (13)(24)],$$

or

$$G_2'' = [1, (14)(23)].$$

It may happen that there exist several decompositions of the same group, as in the last example; in this case, however, *the factors of composition for the different decompositions are the same excepting their order.* For the proof of this theorem we refer to Netto, §81.

44. *The series of composition of the symmetric group of n letters consists of the symmetric group, the alternate group and the group* 1:

$$G_{n!},\ G_{\frac{n!}{2}},\ 1, \tag{1}$$

excepting the only case $n = 4$.

Proof: 1). The alternate group is a *self-conjugate* subgroup of the symmetric group, since (No. 36) the transformed of an even substitution by any substitution is always an even substitution; moreover, it is a *maximum* self-conjugate subgroup, since the index $\nu = 2$ is prime.

2). *The alternate group is simple,* excepting the case $n = 4$; the proof is founded on the following principle: If a self-conjugate subgroup H of a group G contain a substitution h, then it contains also all those substitutions which are conjugate with h under G, say

$$h,\ h',\ h'',\ \ldots .\ h^{(\mu-1)}, \tag{2}$$

and therefore also the group generated by these substitutions,

$$G(h,\ h',\ h'' \ldots .\ h^{(\mu-1)}) \qquad \text{(see No. 19)},$$

which we will denote by H_h.

In the present case G is the alternate group $G_{\frac{n!}{2}}$:

a). Let us first choose for h a circular substitution of three letters, say

$$h = (123),$$

which is sure to be contained in $G_{\frac{n!}{2}}$, since it is an even substitution (No. 16).

The corresponding series (2) *of conjugate substitutions consists of all the circular substitutions of three letters;* that is to say, it is always possible to find in $G_{\frac{n!}{2}}$ a substitution g which transforms (123) into any other circular substitution $(\alpha\beta\gamma)$:

$$(\alpha\beta\gamma) = g^{-1}(123)\,g;$$

namely, one of the two substitutions

$$a = \begin{pmatrix} 1\ 2\ 3\ 4\ 5 \ldots\ldots n \\ \alpha\,\beta\,\gamma\,\delta\,\varepsilon \ldots\ldots \lambda \end{pmatrix}$$

and

$$a' = \begin{pmatrix} 1\ 2\ 3\ 4\ 5 \ldots\ldots n \\ \alpha\,\beta\,\gamma\,\varepsilon\,\delta \ldots\ldots \lambda \end{pmatrix} = a.(\delta\varepsilon),$$

both of which transform $(1\,2\,3)$ into $(\alpha\beta\gamma)$ (No. 34), is sure to be even and may therefore be chosen for g.*

b). *The group H_h, generated by all the circular substitutions of three letters, is the alternate group.*

For the product of any two transpositions can always be written as a product of circular substitutions of three letters; namely, if the two transpositions have one letter in common, we have

$$(\alpha\beta)(\alpha\gamma) = (\alpha\beta\gamma),$$

if they have no letter in common, we have

$$(\alpha\beta)(\gamma\delta) = (\alpha\beta)(\alpha\gamma).(\gamma\alpha)(\gamma\delta) = (\alpha\beta\gamma).(\gamma\alpha\delta).$$

Hence any even substitution can be decomposed into a product of circular substitutions of three letters; and since, conversely, the product of any number of circular substitutions of three letters is always even, our statement is proved and we have $H_h = G_{\frac{n!}{2}}$, if h is a circular substitution of three letters.

c). To this case all other cases can be reduced; namely, if we choose for h any other substitution, it is always possible to form out of the conjugate substitutions (2) a product which is a circular substitution of three letters.

We confine ourselves to the case $n = 5$, referring for the general case to Netto, §84.

The alternate group of five letters, G_{60} contains, besides the circular substitutions of three letters, substitutions of the types $(1\,2)(3\,4)$ and $(1\,2\,3\,4\,5)$.

α). If H contain the substitution $h = (1\,2)(3\,4)$, it contains also the transformed of h by the even substitution $g = (1\,2)(4\,5)$, viz. $h' = (2\,1)(3\,5)$, and consequently also the product $hh' = (3\,4\,5)$; hence $H = G_{60}$ according to *b*).

(This conclusion cannot be reached if $n = 4$, which accounts for the existence of the self-conjugate subgroup G_4 of G_{12}.)

* This conclusion can be reached only when $n > 4$.

β). If H contain the substitution $h = (12345)$, it contains also the transformed of h by the even substitution $g = (23)(45)$, viz. $h' = (13254)$ and consequently also the product $hh' = (153)$; hence again $H = G_{60}$.

Thus the alternate group of five letters is indeed found to be simple.

45. The above decomposition (1) of the symmetric group is the only possible one; or, in other words, *the alternate group is the only maximal self-conjugate subgroup of the symmetric group.* To prove it, we apply the same principle as above: If a self-conjugate subgroup H of the symmetric group contain a substitution h, it contains also all substitutions *similar* with h, since any two similar substitutions are conjugate under the symmetric group.

We confine ourselves again to the case $n = 5$, referring for the general case to Netto, §50.

The even substitutions lead exactly as in No. 44 to the alternate group. The substitution $h = (12)$ leads to the symmetric group, since the latter can be generated by all the transpositions.

Finally, if H contain the substitution $h = (1324)$ or the substitution $(132)(45)$, it contains also their squares, viz. $(12)(34)$ or (123) respectively, and consequently the alternate group.

All different types of substitutions being thus exhausted, the alternate group is the only self-conjugate subgroup of the symmetric group (besides the symmetric group and the group 1).

46. We are now able to answer the question proposed at the end of No. 32; from the theorems of No. 37 and 42 we deduce the result:

In order that it be possible to choose the functions $\xi, \eta, \ldots V$ of No. 31 such that *all the resolvents of the chain* (4) of No. 31 become *binomial equations* of prime degrees, it is necessary and sufficient that

1). All the indices $\lambda, \mu, \ldots \rho$ are *prime numbers.*

2). In the series of groups

$$G_{n!},\ H,\ I \ldots M,\ 1 \tag{1}$$

each group is a *self-conjugate subgroup* of the group immediately preceding it.

Under these conditions this series of groups (1) is precisely what we have defined as a series of composition of the symmetric group, since a self-conjugate subgroup of prime index is always a maximal self-conjugate subgroup, and therefore the above criterion may also be expressed in this form:

The necessary and sufficient condition for a chain of binomial resolvent equations is that all the factors of composition of the symmetric group be prime numbers.

Now these factors are, according to No. 43 and 44,

For $n = 4$: $\qquad 2,\ 3,\ 2,\ 2;$

for any other n: $\qquad 2,\ \frac{n!}{2}.$

Hence the above condition is satisfied for $n = 3$ and $n = 4$, but not for $n > 4$.

However, it would be erroneous to think that with this we have proved the impossibility of solving, by radicals, the general equation of a higher than the fourth degree. To complete our proof we should have to show that the method proposed in No. 31 is *the only possible method*; to this effect we should have to establish the two following propositions due to Abel:

Every equation which is solvable by radicals can be reduced to a chain of binomial equations of prime degrees, and

The roots of these binomial equations are expressible as rational functions of the roots of the given equation (see Netto, §201–210).

I omit, however, to enter here upon the proof of these theorems, since they have been superseded by Galois' theory of the solution of equations by radicals, in connection with which we shall have to reconsider the same questions from a more general point of view in the second part of this memoir.

47. But still I may briefly indicate already here how it happens that certain *special* equations of a higher degree are solvable by radicals, though the general equation be not solvable.

In the case of the general equation the symmetric functions are the only rational functions of the roots which are rationally expressible in terms of the coefficients, and consequently we must, in the series of groups of No. 31, start from the symmetric group.

In the case of special equations, on the contrary, it may happen that besides the symmetric functions some asymmetric function ϕ of the roots is rationally expressible in terms of the coefficients; in this case we need only to start from the group G to which ϕ belongs; and if the factors of composition of this group G are all prime, we can descend by the method of No. 31 from the function ϕ through a chain of binomial equations of prime degrees to an $n!$ valued function V, and the given equation is solvable by radicals.

But before all this can be established rigorously, a number of difficulties have to be removed, which arise the very moment we pass from the general equation to a special equation (see below, No. 55).

§10.—*Groups of Operations.*

48. The notion of a group as defined in No. 13 for the special case of substitutions, may be extended to any operations for which some definition of a product can be given.

To illustrate it, let us consider *the rotations of a rigid body about a fixed point;* let S be a first rotation through a given angle round a given axis, fixed in space and passing through the fixed point, S' another rotation of the same kind; then the successive application of first S, then S' is equivalent to one single rotation which is called the *product* SS' of the two rotations S and S' (see Thomson and Tait, Natural Philosophy, art. 95), and a system of rotations is said to form a *group*, if the product of any two of its rotations belongs itself to the system.

From this definition it is evident that those rotations which leave a regular polyedron congruent with its original position ("mit sich selbst zur Deckung bringen"), form a group.

In the case of a *regular tetraedron*, for example, there are $N = 12$ such rotations: The identical rotation (angle zero);

four rotations through an angle of $\frac{2\pi}{3}$; each round an axis passing through one of the corners and the middlepoint of the opposite face;

four rotations round the same axes, but through an angle of $\frac{4\pi}{3}$; these eight rotations have the period 3.

Finally, three rotations through an angle π round an axis passing through the middlepoints of two opposite edges, having the period 2.

These twelve rotations form a group called the *tetraedron-group*.

49. Most of the definitions and propositions on substitution-groups given in the foregoing paragraphs still hold for groups of any operations. For instance, those rotations of the tetraedron-group G which leave one of its *corners* unaltered —which we shall denote by ψ in order to show the complete analogy with the developments in No. 23 and 35—form a *subgroup H of the tetraedron-group* of

* References concerning §10: Jordan, No. 54–59, 76–81; Netto, §80–84.

index $\nu = 4$; it is a cyclic group consisting of the different powers of one of the rotations of period 3.

If operated upon by all the rotations of G, the corner ψ takes four different positions, viz. it comes to coincide successively with the four corners

$$\psi_1 = \psi, \ \psi_2, \ \psi_3, \ \psi_4$$

of the tetraedron, which for this reason are said to be *conjugate points* with respect to the tetraedron-group.

If g_α denote a rotation of G which brings the corner ψ to the place occupied, before the rotation, by the corner ψ_α, then the corner ψ_α remains unaltered by the subgroup

$$H_\alpha = g_\alpha^{-1} H g_\alpha,$$

which is said to be conjugate with H under G.

When the tetraedron is operated upon by a rotation of G, the four corners undergo a certain substitution; thus we obtain, as in No. 38, a *group* Γ *of substitutions among the letters* $\psi_1, \psi_2, \psi_3, \psi_4$ *isomorphic* with G; and since the rotation 1 is the only one that leaves all four corners at the same time unaltered, Γ is of the same order 12 as G and the isomorphism is *holoedric* (No. 39). Γ is therefore the *alternate group* between the four letters $\psi_1, \psi_2, \psi_3, \psi_4$ (No. 16).

In a similar way the consideration of an *edge* leads to a cyclic subgroup of order 2, and a substitution-group between the six edges, holoedrically isomorphic with G.

The consideration of the *straight line joining the middlepoints of two opposite edges* affords an example of a self-conjugate subgroup. For there are three such lines conjugate under G, all of which remain unaltered by the same subgroup of G, viz. the so-called "Vierergruppe" (Klein), consisting of the rotation 1 and the three rotations of period 2; the "Vierergruppe" is therefore a self-conjugate subgroup of the tetraedron-group.

Analogous considerations can be applied to the other regular polyedra:

The *Octaedron-group* is of order $N = 24$, and is holoedrically isomorphic with the symmetric group of four letters; the cube, being the polar figure of the octaedron, belongs to the same group.

The *Icosaedron-group* is of order $N = 60$, and is holoedrically isomorphic with the alternate group of five letters; the pentagon dodecaedron belongs to the same group.

To these groups may be added the *Cyclic groups* which leave the regular pyramids unaltered, and the *Diedron-groups* which leave the regular double pyramids unaltered (see Klein, Vorlesungen über das Ikosaeder, I, 1).

50. Another example of groups of operations is intimately connected with these groups of rotations. Let us, in fact, circumscribe about the polyedron under consideration, a sphere invariably connected with it: the points of the sphere may be analytically represented by the values of a complex variable z; to this effect we assume a system of rectangular coordinates X, Y, Z, fixed in space, with the origin in the middlepoint of the polyedron, and project the points of the sphere from the point of intersection of the positive Z-axis with the sphere ("North-pole") upon the XY-plane ("Plane of the equator"). X, Y being the coordinates of the projection of a point P of the sphere, the point P is represented by the complex quantity

$$z = X + iY.$$

If, now, by a rotation of the sphere about its middlepoint, the point P is brought into a new position P' represented by the complex quantity z', then z' can be expressed as a linear function of z:

$$z' = \frac{az + b}{cz + d},$$

the coefficients a, b, c, d being constant quantities, independent of the position of the point P (see Klein, l. c. I, 2).

If, then, the *product* of two such linear substitutions is defined as the linear substitution formed by combining them in the usual way, it is easily seen that corresponding to every group of rotations we obtain a group of linear substitutions holoedrically isomorphic with the group of rotations.

In the case of the tetraedron, for example, if we choose the lines joining the middlepoints of the three pairs of opposite edges as the axes of the system of coordinates, we obtain the following group of linear substitutions:

$$z' = \pm z,\ \pm \frac{1}{z},\ \pm i.\frac{z+1}{z-1},\ \pm i.\frac{z-1}{z+1},\ \pm \frac{z+i}{z-i},\ \pm \frac{z-i}{z+i}.$$

51. Thus we find, in connection with each regular polyedron, a series of groups of operations, all holoedrically isomorphic among themselves: the group of rotations, various groups of substitutions (permutations), finally, a group of

linear substitutions; and to these we may further add a group of quaternions, since every rotation may be represented by the versor of a quaternion.

All these groups are so intimately connected among themselves that they appear but as different forms of one and the same group *in abstracto.* (Compare different quantics which can be derived from one another by linear transformation.)

In fact, *two holoedrically isomorphic groups of operations G and Γ have all their most important properties in common.* They are of the same order; they have the same multiplication-table; that is, if in one group, $gg' = g''$, then the corresponding operations γ, γ', γ'' respectively of the other group satisfy the same relation $\gamma\gamma' = \gamma''$. Hence it follows that if any relation exists between any number of operations of G, the same relation exists also between the corresponding operations of Γ. In particular, two corresponding operations are of the same period; further, if g and g' are conjugate operations of G, then the corresponding operations γ and γ' are conjugate under Γ, etc.

To every subgroup H of G corresponds a subgroup H of Γ of the same order; and if H be self-conjugate under G, then also H is self-conjugate under Γ. Hence to a series of composition of G corresponds a series of composition of Γ.

52. Let, now, $a, b, \ldots . k$ denote a system of *generating operations* (No. 19) for the group G; then there exist, in general, numerous relations between them. It is always possible to select in various ways a number of these relations, sufficient and necessary to derive from them the multiplication-table of the group; such a set of relations is called a *system of fundamental relations.*

Example: The symmetric group of three letters can be generated by two substitutions (No. 8):

$$a = (1\,2\,3) \text{ and } c = (2\,3) \tag{1}$$

with the fundamental relations

$$a^3 = 1, \; c^2 = 1, \; (ac)^2 = 1. \tag{2}$$

The group is

$$1, \; a, \; b = a^2, \; c, \; d = ac, \; e = a^2c; \tag{3}$$

the value of any product can be found by means of the relations (2); for instance,

$$de = ac.a^2c,$$

but from (2) follows

$$aca = c, \; cac = a^2,$$

hence

$$de = c.ac = a^3 = b,$$

in accordance with $(1\,3)(1\,2) = (1\,3\,2)$.

Now, if $\alpha, \beta, \ldots . \varkappa$ are the operations of the group Γ which correspond to the operations $a, b, \ldots . k$ respectively, then $\alpha, \beta, \ldots . \varkappa$ form a system of generating operations for the group Γ and satisfy exactly the same set of fundamental relations.

Conversely: Suppose two groups G and Γ are generated by the same number of generating operations $a, b, \ldots . k$ and $\alpha, \beta, \ldots . \varkappa$ respectively, satisfying the same fundamental relations; if, then, we coordinate a with α, b with β and so on, and besides every product

$$a^{\mu} b^{\nu} \ldots . a^{\mu'} b^{\nu'} \ldots .$$

with the similar product

$$\alpha^{\mu} \beta^{\nu} \ldots . \alpha^{\mu'} \beta^{\nu'} \ldots . ,$$

then the correspondence thus established between the two groups is one of holoedric isomorphism.

Hence it follows that the group *in abstracto* of which various holoedrically isomorphic groups are but different forms (No. 51), may be *defined by a system of generating operations with a set of fundamental relations between them.*

Examples: 1). Tetraedron-group:

$$S^3 = 1, \; T^2 = 1, \; (ST)^3 = 1$$

(for instance, $S = (x_2 x_3 x_4)$, $T = (x_1 x_2)(x_3 x_4)$).

2). Octaedron-group:

$$S^4 = 1, \; T^2 = 1, \; (ST)^3 = 1$$

(for instance, $S = (x_1 x_2 x_3 x_4)$, $T = (x_1 x_2)$).

3). Icosaedron-group:

$$S^5 = 1, \; T^2 = 1, \; (ST)^3 = 1.$$

(for instance, $S = (x_1 x_2 x_3 x_4 x_5)$; $T = (x_2 x_3)(x_4 x_5)$).

But a remark is necessary concerning this definition of a group; if we assume at random a set of fundamental relations between a number of generating operations, we will always define a group, but in general the group will

contain an *infinite number of different operations;* for instance, the group generated by two generating operations S, T with the relations

$$S^n = 1; \quad T^2 = 1, \quad (ST)^3 = 1$$

will be finite only if $n \overline{\gtrless} 5$, infinite in all other cases.

53. *Every finite group of operations is holoedrically isomorphic with a regular substitution-group;* this group is given by the multiplication-table of the group. For if

$$S_1, S_2, \ldots . S_N \tag{1}$$

are the operations of the given group G, and S any one of them, then the N products

$$SS_1, SS_2, \ldots . SS_N \tag{2}$$

are a permutation of the operations (1); and if we coordinate with every operation S of G the substitution

$$\sigma = \begin{pmatrix} S_1 & S_2 & \ldots . S_N \\ SS_1 & SS_2 & \ldots . SS_N \end{pmatrix} \tag{3}$$

between the N letters $S_1, S_2, \ldots . S_N$, we obtain a regular substitution-group Γ (No. 24), which is holoedrically isomorphic with the given group G; it is called the *regular form of the given group* G (Dyck).

References concerning groups of operations:

Cayley: On the Theory of Groups. Phil. Mag. 4th Series. Vol. 7 and 18. American Journal of Math., Vol. 1 and 11.

Dyck: Gruppentheoretische Studien, Math. Annalen, Bd. 22.

Klein: Vorlesungen über das Ikosaeder, I, 1.

54. In concluding we give an enumeration of some of the most important classes of groups of operations:

1). Groups of *Substitutions* (permutations).

2). Groups of *Rotations.*

See above, No. 48; groups of *movements* in general have been studied by Schönfliess, Math. Ann. Bd. 28 and 29.

3). Groups of *non-homogeneous linear substitutions* of one variable:

$$z' = \frac{az + b}{cz + d}, \tag{1}$$

the only *finite groups* are those which represent analytically the groups of rotations enumerated in No. 48; see Klein, Ikosaeder, p. 115, where a list of references is given.

The general theory of "discontinuous" *infinite groups* has been developed by Poincaré in a series of memoirs in the Acta Math., Vol. 1, 3, 4, 5.

An important special case is the group of all the linear substitutions of the form (1) in which the coefficients are *integers* satisfying the relation

$$ad - bc = 1.$$

See Dedekind, Borchardt's Journal, Bd. 83; Klein, Ueber die Transformation der elliptischen Functionen, Math. Annalen, Bd. 14.

4). Groups of *homogeneous linear substitutions* of several variables:

$$z'_a = a_{a_1} z_1 + a_{a_2} z_2 + \ldots . \, a_{a_n} z_n. \tag{2}$$

See Jordan, Cours d'analyse III, No. 156; as to finite groups in particular, see Klein, Ikosaeder, p. 123, where a list of references is given.

5). An essentially different class of groups are *Lie's transformation-groups;* such a group is for instance the entire system of all the linear transformations

$$x' = \frac{a_1 x + a_2}{a_3 x + a_4},$$

the coefficients taking all possible values (see Lie, Transformationsgruppen, Math. Ann. Bd. 16, and Lie, Theorie der Transformationsgruppen, Leipzig, 1888).

SECOND PART.

GALOIS' THEORY OF ALGEBRAIC EQUATIONS.

§11.—*Galois' Resolvent.*

55. Hitherto we have always been dealing with the *general equation* of the n^{th} degree; that is to say, we have considered the coefficients $c_1, c_2 \ldots . c_n$, and consequently also the roots $x_1, x_2 \ldots . x_n$, as indeterminate quantities. Accordingly we have regarded two rational functions of the roots as equal only when they were identical for all sets of values of the x's (see No. 7).

But if, on the contrary, the x's are the roots of a given *special equation*, we shall have to consider two rational functions of the roots as equal when their *numerical values* are equal, and it may happen that the numerical value of a rational function ψ of the roots remains unaltered by a substitution which changes the form of the function. For instance, in the case of the equation

$$x^4 + x^3 + x^2 + x + 1 = 0$$

whose roots are the imaginary fifth roots of unity:

$$x_1 = \omega, \quad x_2 = \omega^2, \quad x_3 = \omega^4, \quad x_4 = \omega^3,$$

where $\qquad \omega = e^{\frac{2\pi i}{5}},$

we have $\qquad x_1^2 x_2 = x_3^2 x_1, \text{ viz. } = \omega^4,$

though the two functions $x_1^2 x_2$ and $x_3^2 x_1$ are different in form.

In such a case it would not be allowed to apply Lagrange's theorem to the function ψ because the denominator Δ_ψ vanishes (see No. 28).

Further, those substitutions which leave unaltered the numerical value of a rational function of the roots of a special equation do not, in general, form a group. Thus in the above example the numerical value of the function $x_1^2 x_2$ remains unaltered by the substitutions

$$1; \ (x_3 x_4); \ (x_1 x_3 x_2); \ (x_1 x_3 x_4 x_2),$$

which evidently do not form a group.

In a similar way nearly every one of our theorems on asymmetric functions becomes either entirely wrong, or is true only under certain restricting conditions, if applied to rational functions of the roots of a special equation.

These difficulties which arise in passing from the general equation to a special equation are due to the fact that the so-called general equation is not the true general case, but in truth a very special one; and our next task is to generalize the theorems on asymmetric functions in such a manner that they hold for any given equation without any restrictions (see below No. 67).

56. We consider an equation of the n^{th} degree:

$$f(x) = x^n - c_1 x^{n-1} + c_2 x^{n-2} \dots \pm c_n = 0 \qquad (1)$$

whose coefficients we suppose to be rational functions, with integral coefficients, of a number of *known quantities*, determinate or indeterminate, which we denote by

$$\mathfrak{R}', \ \mathfrak{R}'', \ \mathfrak{R}''' \dots \qquad (2)$$

Any quantity which is expressible as a rational function, with integral coefficients, of these quantities (2), is called *rationally known* or simply *rational.**

By a *rational function* or a *rational relation* we shall always understand a rational function or relation whose coefficients are rational quantities.

---

*The entire system of all these "rational quantities" are said to constitute the "*domain of rationality* ($\mathfrak{R}'$, $\mathfrak{R}''$, ...)" ("Rationalitätsbereich," Kronecker); the simplest domain of rationality consists of all rational numbers; it is called the domain $\mathfrak{R} = 1$.

All interesting applications of Galois' theory come under one of the two following cases:

1). There are only a *finite number* of known quantities (2), and all of them are *algebraic* quantities; that is to say, such an $\mathfrak{R}$ is either an *algebraic number* satisfying an algebraic equation with integral coefficients, or an *indeterminate quantity* or an *algebraic function* satisfying an algebraic equation whose coefficients are rational functions with integral coefficients of some indeterminate quantities.

Examples:

$$x^n + x^{n-1} + \dots + x + 1 = 0; \quad (\mathfrak{R} = 1),$$

$$x^3 - \frac{-1+\sqrt{-7}}{2} x^2 + \frac{-1-\sqrt{-7}}{2} x - 1 = 0; \quad (\mathfrak{R}' = \sqrt{-7}).$$

Modular equation between $x = v$ and $\mathfrak{R}' = u$. Equation of the degree n^2 for the division of $\operatorname{sn} u$:

$$x = \operatorname{sn} \frac{u}{n}; \ \mathfrak{R}' = k^2; \ \mathfrak{R}'' = \operatorname{sn} u; \ \mathfrak{R}''' = \sqrt{(1-\mathfrak{R}''^2)(1-\mathfrak{R}'\mathfrak{R}''^2)}$$

(see Kronecker, Grundzüge einer arithmetischen Theorie der algebraischen Grössen, Borchardt's Journal, Bd. 92; and Molk, Sur la notion de la divisibilité, etc., Acta Math., Vol. 6, pg. 20, 35, 40).

2). *All constant quantities* are considered as being known, and besides a finite number of variable quantities $\mathfrak{R}'$, $\mathfrak{R}''$ .... $\mathfrak{R}^{(\mu)}$, partly independent variables, partly algebraic functions, in the sense of the theory of functions, of some independent variables; we shall denote such a domain by

$$(\text{Const.}, \mathfrak{R}', \mathfrak{R}'' \dots \mathfrak{R}^{(\mu)}).$$

"Two rational functions of the roots are *equal*" will be understood to mean: *a*). If the known quantities (2) are all determinate: "*equal in numerical value.*" *b*). If some of the known quantities are indeterminate: "*equal for all values of these indeterminate quantities.*"

"*Different*" or "*distinct*" means: not equal, in this sense.

Accordingly we shall say a rational function ϕ of the roots *remains unaltered* by a substitution s, if the new function ϕ_s is equal, in this sense, to ϕ.

All these definitions are generalizations of our former ones; in fact, the so-called *general equation* is comprised as a special case in the above assumptions; the known quantities are, in this case, the n indeterminate coefficients

$$\Re', \Re'' \ldots \Re^{(n)}$$

and the equation is then

$$x^n + \Re' x^{n-1} + \Re'' x^{n-2} + \ldots + \Re^{(n)} = 0.$$

57. An integral function $F(x)$ or an equation $F(x) = 0$, whose coefficients are rational quantities, is said to be *reducible* (in the domain $(\Re'\Re'' \ldots)$) if it can be decomposed into factors of a lower degree whose coefficients are themselves rational quantities; *irreducible* if no such decomposition is possible.*

A reducible integral function can always, and only in one way, be decomposed into irreducible factors (see Kronecker, l. c. §§1–4, and Molk, l. c. Chap. II).

If one root of an irreducible equation $F(x) = 0$ satisfies at the same time another equation $G(x) = 0$ whose coefficients are likewise rational quantities, then ALL *the roots of the irreducible equation satisfy the second equation* (see Serret, No. 100).

Hence, if two irreducible equations have one root in common, they are altogether identical.

58. To the assumptions of No. 56 concerning the nature of the coefficients of the given equation

$$f(x) = 0, \tag{1}$$

we add one more hypothesis, viz. that *its n roots $x_1, x_2, \ldots x_n$ are all distinct.*

---

* In this definition we include expressly the case where some of the quantities $\Re', \Re'' \ldots$ do not explicitly enter into the coefficients; the notion of irreducibility is therefore *relative.* For instance, the equation

$$x^4 + x^3 + x^2 + x + 1 = 0$$

is irreducible in the domain $\Re = 1$; but the coefficients may as well be considered as belonging to the domain $(\sqrt{5})$, and in this case the equation is reducible, since it may be written

$$\left(x^2 + \frac{1+\sqrt{5}}{2}x + 1\right)\left(x^2 + \frac{1-\sqrt{5}}{2}x + 1\right) = 0.$$

a). Under these conditions *it is always possible to construct a rational function of the roots which takes $n!$ different values* if operated upon by all the $n!$ substitutions between the letters $x_1, x_2, \ldots x_n$.

Such a function is, for instance,

$$V_1 = m_1x_1 + m_2x_2 + \ldots . m_nx_n, \tag{3}$$

if the coefficients m be properly chosen.

For two values V_a and V_b derived from V_1 by two different substitutions a and b respectively, cannot be equal for all values of the parameters m, since all the roots x are supposed to be distinct. Hence it is always possible to choose for the m's such rational quantities which satisfy none of the $\frac{n!\,(n!-1)}{2}$ relations of the form $V_a = V_b$; V_1 is then in fact an $n!$-valued function.

From an equation of the form

$$V_{a'} = V_a$$

we may then always infer $a' = a$.

b). *It is always possible to construct a rational function of the roots of* (1) *which remains unaltered by the substitutions of any given substitution-group*

$$H = [1,\ a,\ b, \ldots . k]$$

between the letters $x_1, x_2, \ldots x_n$.

Such a function is for instance

$$\psi = (r - V_1)(r - V_a)(r - V_b) \ldots . (r - V_k),$$

r being a properly chosen rational quantity (compare No. 18, and Jordan, No. 351).

59. *Any rational function of the roots of the given equation* (1) *is rationally expressible in terms of such an $n!$-valued function V_1*:

$$\phi\,(x_1, x_2, \ldots . x_n) = \Phi\,(V_1); \tag{4}$$

moreover, *the function ϕ_a derived from ϕ* by any substitution a between the letters $x_1, x_2, \ldots . x_n$, *is the same function Φ of V_a*:

$$\phi_a\,(x_1, x_2, \ldots . x_n) = \Phi\,(V_a). \tag{5}$$

Proof: We consider first the x's as indeterminate quantities; in this case (4) follows from Lagrange's theorem (No. 27) and (5) is a self-evident consequence of (4). Afterwards we replace the indeterminate quantities x by the

roots of the given equation, which we may safely do, since the discriminant Δ_{V_1} is not zero (compare No. 28); the coefficients of the rational function Φ are then changed into rational quantities of our domain $(\mathfrak{R}', \mathfrak{R}'' \ldots .)$*

In particular, *the roots of the given equation are rationally expressible in terms of* V_1:

$$x_1 = \psi_1(V_1), \ x_2 = \psi_2(V_1) \ldots . x_n = \psi_n(V_1). \tag{6}$$

60. The $n!$ conjugate values of the function V_1 are the roots of a resolvent equation of the degree $n!$ (No. 26),

$$F(V) = 0, \tag{7}$$

whose coefficients are rational quantities. Let now $F_0(V)$ be that irreducible factor† of $F(V)$ which has the root $V = V_1$; *the irreducible equation*

$$F_0(V) = 0 \tag{8}$$

is then called Galois' Resolvent of the given equation (1).

Let $N(\overline{\gtrless} n!)$ be its degree. Its roots are contained among the roots of (7) and are therefore some of the conjugate values of V_1; hence they are rationally expressible in terms of V_1, according to No. 59; thus we have the theorem:

All the roots of Galois' resolvent are rationally expressible in terms of one of them.

The solution of the given equation (1) is equivalent to the determination of one root V_1 of Galois' resolvent, since all the roots of (1) are rationally expressible in terms of V_1 (*Galois' Principle;* compare No. 31).

§12.—*Group of an Equation.*

61. Let now

$$V_1, \ V_a, \ V_b, \ldots . V_l \tag{9}$$

be the N roots of Galois' resolvent (8), derived from V_1 by the substitutions

$$1, \ a, \ b, \ldots . \ l \tag{10}$$

respectively.

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*The second part of the theorem is only true if no reductions are made in the expression of the rational function Φ_1 (by means of the irreducible equation $F_0(V_1) = 0$ of No. 60), after the indeterminate x's have been replaced by the roots of (1).

† With respect to the domain of rationality $(\mathfrak{R}', \mathfrak{R}'' \ldots .)$. If $F(V)$ itself should be irreducible, we have simply $F_0(V) = F(V)$.

These substitutions form a group; to prove it we have but to show that if V_a and V_b are any two roots of (8), then also V_{ab}, derived from V_1 by the product ab, will be a root of (8).

V_a being a root of (8), we have

$$F_0(V_a) = 0.$$

Now, according to No. 60, V_a is expressible as a rational function of V_1, say

$$V_a = \theta(V_1), \tag{11}$$

therefore we have $$F_0[\theta(V_1)] = 0.$$

The root V_1 of the irreducible equation (8) satisfies therefore at the same time the equation

$$F_0[\theta(V)] = 0, \tag{12}$$

whose coefficients are rational quantities, consequently (No. 57) all the roots of (8) satisfy (12), in particular also the root V_b; that is,

$$F_0[\theta(V_b)] = 0.$$

But from (11) follows, according to the lemma of No. 59,

$$V_{ab} = \theta(V_b); \tag{13}$$

hence V_{ab} is also a root of Galois' resolvent (8), and consequently the product ab is itself one of the substitutions (10). Thus we have proved the theorem:

The N substitutions by which the N roots of Galois' resolvent can be derived from one of them, V_1, form a group.

This group is called *the group of the given equation* (1) *with respect to the domain of rationality* $(\mathfrak{R}', \mathfrak{R}'', \ldots)$;* we denote it by G.

62. The group G of the given equation (1) possesses the following two fundamental properties:

A. *Every rational function of the roots of $f(x) = 0$ which remains unaltered by all the substitutions of the group G, is rationally known.*

B. *Every rational function of the roots of $f(x) = 0$ which is rationally known remains unaltered by all the substitutions of G.*

* It may, however, be remarked that this theorem as well as the lemmas of No. 59 can be proved directly without recurring to the case of indeterminate x's, and this would indeed be more in accordance with the spirit of Galois' theory. The proofs may be found in Serret, No. 502, 504.

Proof: Let $\phi = \phi(x_1, x_2, \ldots . x_n)$ be a rational function of the roots; we express it in terms of V_1:

$$\left.\begin{aligned} &\phi = \Phi(V_1); \\ &\text{according to the lemma (5) of No. 59, we have then} \\ &\phi_a = \Phi(V_a), \\ &\phi_b = \Phi(V_b), \\ &\ldots\ldots\ldots\ldots \\ &\phi_l = \Phi(V_l), \end{aligned}\right\} \tag{14}$$

Φ denoting throughout the same rational function.

Let us now first suppose, as in A, that

$$\phi = \phi_a = \phi_b = \ldots . = \phi_l;$$

then we obtain, by adding the equations (14),

$$\phi = \frac{1}{N}\left[\Phi(V_1) + \Phi(V_a) + \Phi(V_b) \ldots . + \Phi(V_l)\right].$$

The expression on the right-hand side is a symmetric function of all the N roots of Galois' resolvent

$$F_0(V) = 0, \tag{8}$$

and therefore rationally expressible in terms of the coefficients of this equation, which, in their turn, are rational quantities. Hence ϕ itself is rationally known.

Q. E. D.

To prove the conversion B, let us suppose that ϕ is rationally known,

$$\phi(x_1, x_2, \ldots . x_n) = \text{Rat.}(\mathfrak{R}', \mathfrak{R}'', \ldots .) = r,$$

On account of (14) this relation may be written

$$\Phi(V_1) - r = 0.$$

One root of the irreducible equation (8), viz. $V = V_1$, satisfies therefore the equation

$$\Phi(V) - r = 0, \tag{15}$$

whose coefficients are rational quantities; hence (No. 57) all the roots of (8) must satisfy (15); that is,

$$\Phi(V_1) = \Phi(V_a) = \Phi(V_b) \ldots . = \Phi(V_l) = r,$$

or on account of (14),

$$\phi = \phi_a = \phi_b \ldots\ldots = \phi_i = r;$$

that is, ϕ remains unaltered by all the substitutions of G. Q. E. D.

Remark: The proposition B may also be expressed in the following form, which is preferable in most applications:

B'. *Any rational relation between the roots remains true if operated upon by any substitution of the group G.*

63. *These two properties are characteristic of the group of an equation;* that is to say, if the two propositions A and B hold for a group G', then G' is identical with G.

a). Let us first suppose we know of the group

$$G' = [1,\ a',\ b' \ldots\ldots m']$$

that every rational function of the roots of (1) which remains unaltered by all the substitutions of G' is rationally known.

We form the function

$$(V - V_1)(V - V_{a'}) \ldots\ldots (V - V_{m'}) = F'(V);$$

its coefficients being symmetric functions of $V_1,\ V_{a'} \ldots\ldots V_{m'}$, remain unaltered by G' and are therefore, according to our hypothesis, rational quantities. Hence follows (No. 57) that $F'(V)$ is divisible by the irreducible function $F_0(V)$ and therefore the roots of $F_0(V)$ must all be contained among the roots of $F'(V)$; hence we have the result:

If for a group G' the proposition A holds, then G' must contain the group G of the equation, or

The group of the equation is the smallest group for which the proposition A *holds.*

b). Let us now suppose we know of the group

$$G'' = [1,\ a'',\ b'' \ldots\ldots p'']$$

that every rational relation between the roots remains true if operated upon by all the substitutions of G''. Applying this hypothesis to the relation $F_0(V_1) = 0$ we obtain

$$F_0(V_{a''}) = 0,\quad F_0(V_{b''}) = 0 \ldots\ldots F_0(V_{p''}) = 0;$$

consequently the values

$$V_1,\quad V_{a''},\quad V_{b''} \ldots\ldots V_{p''}$$

must be contained among the roots of $F_0(V) = 0$; hence

If for a group G'' the proposition B holds, then G'' must be contained in G, or

The group of an equation is the largest group for which the proposition B *holds.*

c). Now, if for the same group, G', the two propositions A and B hold, then G' must at the same time contain G and be contained in G, which is only possible if $G' = G$. Q. E. D.

Hence the two fundamental properties may be used as a *new definition* of the group of an equation; it has the advantage of showing at once that the group of an equation is independent of the choice of the $n!$-valued function V_1.

64. If the coefficients of the given equation contain some indeterminate parameters, the group of the equation may be defined still in an entirely different way.

Let us, in order to fix the ideas, consider an equation of the n^{th} degree

$$f(x;\, t) = 0, \tag{1}$$

whose coefficients are rational functions, with any constant coefficients, of one indeterminate parameter t.

The n roots $x_1, x_2, \ldots x_n$ of (1) are algebraic functions of t. If, then, the parameter t, which we represent as usual by a point in a t-plane, describes a closed curve in its plane, starting from some fixed initial position, the n roots are only interchanged among themselves. Hence if the point t describes all possible closed curves, the letters $x_1, x_2, \ldots x_n$ undergo a series of different substitutions, which are easily seen to form a group, called the *monodromy-group* of the equation (1) with respect to the parameter t. Now, this group can be proved to be identical with the group of the equation (1), as defined in No. 61, with respect to the domain of rationality (Const., t) which consists (see No. 56, note) of all the rational functions with any constant coefficients, of the parameter t (see Jordan, No. 390).

If, in particular, the coefficients of (1) are rational functions, with *integral* coefficients of t, there exist at the same time a monodromy-group and an "algebraic group" of the equation (1), the former referring to the domain of rationality (Const., t), the latter to the domain (t).

In this case *the monodromy-group is always a self-conjugate subgroup of the algebraic group* (see Jordan, No. 391).

Moreover, there exists always an algebraic number ε such that the monodromy-group is identical with the group of (1) with respect to the domain (ε, t) (see Jordan, No. 391).

Example:

$$f(x; t) = x^n - t = 0 \qquad (n \text{ prime}).$$

The n roots may be written

$$x_0, \ x_1 = \omega x_0, \ x_2 = \omega^2 x_0, \ldots . x_{n-1} = \omega^{n-1} x_0,$$

ω denoting a primitive n^{th} root of unity; the monodromy-group is the cyclic group consisting of the n different powers of the substitution

$$(x_0, x_1, x_2, \ldots . x_{n-1}).$$

The algebraic group is the so-called "metacyclic group" (see below, No. 79), which, in fact, contains the cyclic group as a self-conjugate subgroup.

The quantity ε is in this case ω; in the domain (ω, t) the equation is, in fact, a "cyclic equation" (see below, No. 73).

65. *If an equation is irreducible, its group is transitive, and vice versa.* For if the group G of the equation $f(x) = 0$ is intransitive (No. 24), connecting transitively only the roots $x_1, x_2, \ldots . x_m$ $(m < n)$, then any symmetric function of these m roots remains unaltered by all the substitutions of G and is, consequently, rationally known (No. 62, A); the product $(x - x_1)(x - x_2) \ldots . (x - x_m)$ is therefore a rational divisor of $f(x)$ and $f(x)$ is reducible.

Conversely, if $f(x)$ is reducible and $f_0(x) = (x - x_1)(x - x_2) \ldots . (x - x_m)$ a rational factor of $f(x)$, then the rational relation $f_0(x_1) = 0$ remains true if operated upon by any substitution of G (No. 62, B′); therefore no substitution of G can replace x_1 by one of the roots $x_{m+1}, \ldots . x_n$; that is to say, G is intransitive.

Thus both parts of our proposition are proved.*

§13.—*Theorems on Asymmetric Functions of the Roots.*

66. In the case of the so-called "general" equation

$$x^n + \mathfrak{R}' x^{n-1} + \mathfrak{R}'' x^{n-2} + \ldots . + \mathfrak{R}^{(n)} = 0,$$

the $\mathfrak{R}$'s being indeterminate quantities (see No. 56), the symmetric functions are the only rational functions of the roots which are rationally known.

* References concerning §11 and 12: Galois, Journal de mathématiques pures et appliquées, Vol. XI (1846); Serret, No. 502, 504, 577, 578, 579; Jordan, No. 348-357; Netto, §221, §228; Bachmann, Ueber Galois' Theorie der algebraischen Gleichungen, Math. Ann. 18.

For in any relation of the form

$$\phi(x_1, x_2, \ldots . x_n) = \text{Rat.}\,(\mathfrak{R}', \mathfrak{R}'', \ldots . \mathfrak{R}^{(n)}),$$

the right-hand side can be transformed into a symmetric function of $x_1, x_2, \ldots x_n$, and since the relation is understood to hold for all values of $\mathfrak{R}', \mathfrak{R}'', \ldots . \mathfrak{R}^{(n)}$ (see No. 56, definition of "equal"), and consequently also for all values of $x_1, x_2, \ldots . x_n$, ϕ must itself be a symmetric function.

Hence follows, according to No. 63, a), that *the group of the "general" equation is the symmetric group.*

Hence the fundamental theorem A (No. 62), if applied to the "general" equation, assumes the familiar form:

Every symmetric function of the roots is rationally expressible in terms of the coëfficients.

67. In a similar way the theorems on asymmetric functions of n indeterminate quantities of §§3, 4 and 5 may be considered as special cases of more general theorems which hold for the roots of any given equation and in which the group of the equation takes the place of the symmetric group in the former case.

a). Thus the theorem of No. 13:

"Those substitutions (of the symmetric group) which leave a rational function of n indeterminate quantities $x_1, x_2, \ldots . x_n$ unaltered, form a group," is a special case of the following:

Those substitutions OF THE GROUP G *of the equation* $f(x) = 0$ *which leave a rational function of its roots unaltered, form a group.*

We may repeat, word by word, the proof of No. 13, confining ourselves, however, exclusively to the consideration of substitutions of the group G, and replacing the truism:

"Every rational relation between the indeterminate quantities $x_1, x_2, \ldots x_n$ remains true if operated upon by any substitution," by the corresponding general theorem B′:

"Every rational relation between the roots of $f(x) = 0$ remains true if operated upon by any substitution *of the group* G of the equation."

Henceforward we shall say, the function ψ *belongs* to the group H (subgroup of G), if it remains unaltered by all the substitutions of H and by no other substitutions of G.

b). Modifying in the same way the conclusions of No. 23 and No. 26, and

replacing the fundamental theorem on symmetric functions by the fundamental theorem A (No. 62), we find:

If ν be the index of H under G, then ψ takes ν different values

$$\psi_1 = \psi, \quad \psi_2, \ldots . \psi_\nu$$

if operated upon by all the substitutions of G.

These ν values are the roots of a resolvent equation of the ν^{th} degree

$$g(\psi) = 0$$

whose coefficients are rational quantities.

c). Further, *Lagrange's theorem* takes the form

If a rational function ϕ of the roots of $f(x) = 0$ remains unaltered by all those substitutions of the group G which leave another function ψ unaltered, then ϕ is rationally expressible in terms of ψ:

$$\phi = \text{Rat.}\,(\psi);$$

and the theorem *is true without exception*, for the denominator of Rat. (ψ) is now the square of the product of all the differences of the ν conjugate values of ψ *under G*, and is therefore not zero according to *b*) (compare No. 28).

Remark: In all these generalized theorems, the expressions "equal," "different," "unaltered" are understood in the sense of No. 56.

68. *Application to regular equations.* Let us consider an equation of the n^{th} degree

$$f(x) = 0 \tag{1}$$

whose group G is regular, that is (No. 24), transitive and of order n; such an equation is called a *regular equation.*

a). Since the group is supposed to be transitive, the equation (1) is *irreducible* (No. 65).

b). Since, besides, the group is of order n, the substitution 1 is the only substitution of G which leaves x_1 (or any other root) unaltered; x_1 "belongs" therefore to the group 1 (No. 67, *a*), and consequently *all the roots are rationally expressible in terms of any one of them*, for instance x_1:

$$x_2 = \theta_2(x_1), \quad x_3 = \theta_3(x_1) \ldots . x_n = \theta_n(x_1). \tag{2}$$

These two properties are *characteristic* of a regular equation.

For the irreducibility implies the *transitivity* of the group G; and from the relations (2) follows, with the aid of the fundamental theorem B′ (No. 62), that

the substitution 1 is the only substitution of G which does not displace the letter x_1; hence, according to No. 24, the group G is indeed regular.

§14.—*Reduction of the Group by Adjunction.*

69. In all the preceding investigations we have been "operating in the domain of rationality,"

$$\mathfrak{R} = (\mathfrak{R}', \mathfrak{R}'', \ldots.),$$

to which the coefficients of our given equation

$$f(x) = 0 \tag{1}$$

were supposed to belong, and the definition of irreducibility and accordingly of the group G of the equation referred to this domain $\mathfrak{R}$.

Let us now suppose we had found, by solving some auxiliary equation, an irrational function ξ of the quantities $\mathfrak{R}', \mathfrak{R}'', \ldots.$; we may then, henceforth, consider also ξ as a known quantity, or, in the language of Galois and Kronecker, *adjoin it to our domain of rationality*, so that we obtain the enlarged domain:

$$R' = (\xi;\ \mathfrak{R}', \mathfrak{R}'', \ldots.).$$

The coefficients of (1) belong *a fortiori* also to the enlarged domain R' and we may therefore repeat all our former developments with respect to the domain R'.

The group G' of (1) with respect to the new domain is now defined by that factor $F_0'(V;\xi)$ of $F(V)$, rational and irreducible in the domain R', which has the root V_1 (No. 61). Since the function $F_0(V)$ of No. 61 and this irreducible function $F_0'(V;\xi)$ have the root V_1 in common, $F_0(V)$ must be divisible by $F_0'(V;\xi)$ (No. 57), and consequently G' must be a subgroup of G; in Galois' terms: "*by the adjunction of ξ, the group G of (1) is reduced to the subgroup G'*" (including, however, the case $G' = G$, which happens if $F_0(V)$ remains irreducible after the adjunction of ξ, so that $F_0'(V;\xi) = F_0(V)$; compare No. 57, note).

70. We consider first the case where the adjoined quantity is a rational function of the roots:

By the adjunction of a rational function ψ of the roots $x_1, x_2, \ldots. x_n$, which belongs to a subgroup H of G, the group G of the equation is reduced precisely to the subgroup H.

For any rational function ϕ of the roots which remains unaltered by all

the substitutions of H is rationally expressible in terms of ψ according to Lagrange's theorem generalized (No. 67, c) and belongs therefore to the new domain

$$R' = (\psi;\ \mathfrak{R}',\ \mathfrak{R}'',\ \ldots\ldots).$$

Conversely, any rational function ϕ of the roots which is rationally known in the domain R', remains unaltered by all the substitutions of H.

For the relation

$$\phi(x_1, x_2, \ldots\ldots x_n) = \text{Rat.}\,(\psi(x_1, x_2, \ldots\ldots x_n);\ \mathfrak{R}',\ \mathfrak{R}'',\ \ldots\ldots)$$

remains true if operated upon by any substitution of G (No. 62, B') and, *a fortiori*, of its subgroup H; but the substitutions of H leave ψ unaltered, and consequently, the whole right-hand side; therefore also ϕ remains unaltered by all the substitutions of H.

The group H satisfies therefore the two characteristic conditions of the group of (1) with respect to the domain R'.

71. The quantity ψ, which we have considered as known in the last No., is obtained by solving the resolvent equation (No. 67, *b*),

$$g(\psi) = 0$$

of the degree ν, if we denote again by ν the index of the subgroup H under G. *The group of this resolvent equation* (whose roots are the ν conjugate values $\psi_1, \psi_2, \ldots\ldots \psi_\nu$ of ψ under G) *is the group* Γ, *isomorphic with* G, *of those substitutions which the* ψ's *undergo when the* x's *are operated upon by all the substitutions of* G (see No. 38).

For any rational function $R(\psi_1, \psi_2, \ldots\ldots \psi_\nu)$ of the ψ's may at the same time be considered as a rational function of the x's:

$$R(\psi_1, \psi_2, \ldots\ldots \psi_\nu) = r(x_1, x_2, \ldots\ldots x_n);$$

and if R remains unaltered by all the substitutions of Γ, then r remains unaltered by all the substitutions of G, and vice versa. Hence the group Γ can easily be proved to possess the two fundamental properties A and B (No. 62), characteristic of the group of the equation $g(\psi) = 0$.

Corollaries: 1. Since the root ψ_1 can be changed by substitutions of G, and consequently also by substitutions of Γ into every one of the ν values $\psi_1, \psi_2, \ldots \psi_\nu$, Γ is transitive, and therefore *the resolvent* $g(\psi) = 0$ *is irreducible* (No. 65).

2). If the group H to which ψ "belongs" is a *self-conjugate* subgroup of G, then the resolvent is a *regular equation** (No. 39).

3). If, in particular, H is a *maximum self-conjugate* subgroup of G, then the resolvent is a *regular and simple equation* (No. 39, b; 38, 43).

4). Finally, if H is a *self-conjugate* subgroup of *prime* index ν of G, then the resolvent is a *cyclic equation of prime degree* ν (No. 39, c).

5). Let us finally consider the case where the given equation $f(x) = 0$ is *irreducible* and the rational function ψ *contains only one single root:*

$$\psi = \psi(x_1).$$

In this case the group of (1) is transitive (No. 65), and the index of that subgroup I of G which does not displace x_1 is n (No. 24).

Now that subgroup H of G to which ψ belongs contains evidently the group I, and consequently the index ν of H under G, and therefore also *the degree of the irreducible resolvent equation* $g(\psi) = 0$, *is a divisor of* n.

72. An equation $f(x) = 0$ whose group G is composite (No. 43) is called a *composite equation.*

The solution of a composite equation can always be reduced to the solution of a chain of SIMPLE REGULAR *equations.*

For let
$$G - H - I \dots . M - 1$$

be a series of composition (No. 43) of the group G of the given equation; $\lambda, \mu, \dots \rho$ the factors of composition and $\phi, \psi, \dots . \chi, V$ rational functions of the roots belonging to the groups $H, I, \dots . M, 1$ respectively (see No. 58, b). Then ϕ is found by solving a resolvent equation of the degree λ, simple and regular with respect to the original domain (No. 71, 3).

By the adjunction of ϕ, the group G of the equation is reduced to the subgroup H (No. 70).

A second resolvent equation of the degree μ, simple and regular with respect to the new domain furnishes ψ, by the adjunction of ψ the group H of the equation is reduced to the subgroup I and so forth, and our proposition is proved.†

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* The expressions "regular," "simple," "cyclic," etc., are transferred from the group Γ to the equation $g(\psi) = 0$; a regular equation is an equation whose group is regular, etc.

† References concerning §14: Jordan, No. 362, 372; Serret, No. 580, 583; Netto, §228.

§15.—*Cyclic or Abelian Equations.*

73. The equation
$$f(x) = 0 \tag{1}$$
is called a *cyclic equation* if its group G is the cyclic group consisting of the n different powers of a circular substitution, say
$$G = [1, s, s^2, \ldots . s^{n-1}],$$
$$s = (x_0, x_1, \ldots . x_{n-1}),$$
$x_0, x_1, \ldots . x_{n-1}$ denoting the n roots of (1).

The cyclic group G is regular; hence the equation (1) is *irreducible* and all its roots are rationally expressible in terms of one of them (No. 68), say, in particular, $x_1 = \theta(x_0)$.

This rational relation remains true if operated upon by all the substitutions of G (No. 62, B′), hence we obtain
$$x_1 = \theta(x_0),\ x_2 = \theta(x_1),\ x_3 = \theta(x_2) \ldots . x_{n-1} = \theta(x_{n-2}),\ x_0 = \theta(x_{n-1}), \tag{2}$$
or, if we denote
$$\theta[\theta(x_0)] = \theta^2(x_0),\ \theta[\theta^2(x_0)] = \theta^3(x_0), \text{ and so forth,}$$
$$x_1 = \theta(x_0),\ x_2 = \theta^2(x_0) \ldots . x_{n-1} = \theta^{n-1}(x_0),\ x_0 = \theta^n(x_0). \tag{3}$$
Hence the rational function $\theta(x_0)$ has the "period" n and the n *roots "form one cycle."* Now an irreducible equation whose roots are rationally expressible in terms of one of them and form one cycle like (3) is called an *Abelian* equation*, thus we have the result:

Every cyclic equation is an Abelian equation.

74. Let us now, conversely, determine the group G of an irreducible equation $f(x) = 0$, whose n roots satisfy the relation (3). We may unite these relations in one formula
$$x_z = \theta^z(x_0) \qquad (z = 0, 1, 2 \ldots .), \tag{4}$$
if we agree to consider two indices which are congruent (mod n) as equivalent, so that $x_z, x_{z+n}, x_{z+2n} \ldots .$ represent indifferently the same root.

Let now
$$t = \begin{pmatrix} x_0 \ldots . x_z \ldots . \\ x_\alpha \ldots . x_{z'} \ldots . \end{pmatrix}$$

*Properly, *uniserial Abelian* ("einfache Abel'sche," Kronecker); no mistake can arise from this abbreviation, as I shall confine myself in this paper to the consideration of uniserial Abelian equations.

be any substitution of the unknown group G of (1); the relation (4) must then remain true if operated upon by t, hence we have

$$x_{z'} = \theta^{z}(x_a);$$

but according to (3),

$$\theta^{z}(x_a) = \theta^{z}[\theta^{a}(x_0)] = \theta^{z+a}(x_0) = x_{z+a};$$

therefore $x_{z'} = x_{z+a}$ and consequently

$$z' \equiv z + \alpha \qquad (\text{mod } n).$$

The substitutions of G are therefore all of the form

$$\begin{pmatrix} x_0 & x_1 & x_2 & \dots \\ x_a & x_{a+1} & x_{a+2} & \dots \end{pmatrix},$$

or, as we may write

$$\begin{pmatrix} x_z \\ x_{z+a} \end{pmatrix} \qquad (\text{mod } n), \tag{5}$$

or $|z \;\; z+\alpha|(\text{mod } n)$ in Jordan's notation.

The substitution $\begin{pmatrix} x_z \\ x_{z+a} \end{pmatrix}$ is the α^{th} power of the circular substitution

$$s = \begin{pmatrix} x_z \\ x_{z+1} \end{pmatrix} = (x_0, x_1, \dots x_{n-1}),$$

and therefore the group G must be either the cyclic group consisting of the n different powers of s or else one of its subgroups.

But since (1) is supposed to be irreducible, its group must be transitive, and therefore the latter case is to be excluded, since all the subgroups of the cyclic group are intransitive. Thus we find the result:

The group of an abelian equation of the n^{th} degree is a cyclic group of order n.

The terms "cyclic equation" and "abelian" equation are therefore equivalent.

75. If the degree n is prime, the cyclic group G is simple; if, on the contrary, n is composite, let p be any prime factor of n: $n = p.n'$.

If, then, we put $s^p = s'$, the group

$$H = [1, s', s'^2, \dots s'^{n'-1}]$$

is a subgroup of G of prime index p and is moreover self-conjugate under G, since $s^{-\beta}.s^{ap}.s^{\beta} = s^{ap}$ (see No. 36). In decomposing G we may therefore use H as the second group of a series of composition (No. 43). Repeating the same

conclusion with the group H we find: *The factors of composition of a cyclic group of order n are the prime factors of n*, each one repeated as many times as it is contained in n.

Combining this with the results of No. 71, 4 and No. 72, we have the theorem:

An abelian equation of composite degree n can be reduced to a chain of abelian equations whose degrees are the prime factors of n.

76. *An abelian equation of prime degree is always solvable by radicals.*

Proof: Let us adjoin to our domain of rationality an imaginary n^{th} root of unity, ω, and consider the then rational function of the roots of (1):

$$V = x_0 + \omega x_1 + \omega^2 x_2 + \dots \omega^{n-1} x_{n-1}, \tag{6}$$

which is called *Lagrange's expression*, and is usually denoted by (ω, x_0). If operated upon by the circular substitution

$$s = (x_0, x_1, x_2, \dots x_{n-1}),$$

V is changed into $V_s = \omega^{-1} V$, and consequently the n^{th} power $V^n = (\omega, x_0)^n$ remains unaltered by all the substitutions of G and *a fortiori* also by all the substitutions of the group G' of (1) after the adjunction of ω (see No. 69); it is therefore (No. 62, A) rationally known:

$$V^n = \text{Rat.}(\omega, \mathfrak{R}', \mathfrak{R}'', \dots) = r(\omega). \tag{7}$$

But the function V belongs to the group 1, and therefore the roots of (1) are rationally expressible in terms of $V = \sqrt[n]{r(\omega)}$, according to Lagrange's theorem:

$$\begin{gathered} x_\alpha = \Psi_\alpha(V, \omega), \\ \alpha = 0, 1, \dots n-1. \end{gathered} \tag{8}$$

Hence the equation (1) is, in fact, solvable by radicals, provided the n^{th} root of unity ω be known.

The expression of x_α in terms of V can be found in a very elegant way by the following method due to Lagrange: Evidently also the function $(\omega^\kappa, x_0)^n$ is rationally expressible in terms of ω, say

$$(\omega^\kappa, x_0)^n = r_\kappa(\omega).^*$$

*It can be proved that $r_\kappa(\omega) = r(\omega_\kappa)$.

Forming this equation for $\varkappa = 1, 2, \ldots n-1$ and extracting the n^{th} root, we obtain

$$\begin{aligned} x_0 + \omega x_1 + \ldots + \omega^{n-1}x_{n-1} &= \sqrt[n]{r_1(\omega)}, \\ x_0 + \omega^2 x_1 + \ldots + \omega^{2(n-1)}x_{n-1} &= \sqrt[n]{r_2(\omega)}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots \\ x_0 + \omega^{n-1}x_1 + \ldots + \omega^{(n-1)^2}x_{n-1} &= \sqrt[n]{r_{n-1}(\omega)}. \end{aligned}$$

Combining these equations with the following:

$$x_0 + x_1 + \ldots + x_{n-1} = c_1,$$

we have a system of n linear equations, from which we deduce, by multiplying the first by $\omega^{-\alpha}$, the second by $\omega^{-2\alpha}$, and so on, and adding,

$$x_\alpha = \frac{c_1 + \omega^{-\alpha}\sqrt[n]{r_1(\omega)} + \omega^{-2\alpha}\sqrt[n]{r_2(\omega)} \ldots + \omega^{-(n-1)\alpha}\sqrt[n]{r_{n-1}(\omega)}}{n},$$

$$\alpha = 1, 2 \ldots n.$$

The value of one of the $n-1$ radicals which enter into these expressions, for instance $\sqrt[n]{r_1(\omega)}$, may be chosen arbitrarily; the others are then completely determined, since they are rationally expressible in terms of the first; in fact, the quotient

$$\frac{\sqrt[n]{r_\varkappa(\omega)}}{\sqrt[n]{r_1(\omega)}} = \frac{(\omega^\varkappa, x_0)}{(\omega, x_0)^\varkappa}$$

remains unaltered by all the substitutions of G, and is therefore rationally expressible in terms of ω.

77. To obtain the primitive n^{th} root of unity ω we have to solve the equation

$$x^{n-1} + x^{n-2} + \ldots + x + 1 = 0, \tag{9}$$

called *the equation for the division of the circle* in n equal parts, whose roots are $\omega, \omega^2, \omega^3, \ldots \omega^{n-1}$.

This equation is itself an *abelian* equation with respect to the domain $\mathfrak{R} = 1$, if n is prime, as we suppose it to be.

For, in this case, the equation (9) is in the first place irreducible (proofs by Gauss, Eisenstein, Kronecker and others; see Jordan, No. 413, 414; Serret, No. 110; Netto, §160).

Further, all the roots are rationally expressible in terms of one of them and can moreover be arranged in one cycle; for if g denote a primitive root of the prime number n, the roots of (9) may be written

$$x_0 = \omega,\ x_1 = \omega^g,\ x_2 = \omega^{g^2}, \ldots x_{n-2} = \omega^{g^{n-2}},$$

or if we write $x_0^\theta = \theta(x_0)$:

$$x_1 = \theta(x_0),\ x_2 = \theta^2(x_0) \ldots . x_{n-2} = \theta^{n-2}(x_0),\ x_0 = \theta^{n-1}(x_0);$$

the equation (9) is therefore (No. 74) in fact an abelian equation.

According to No. 75, it can be resolved into a chain of abelian equations whose degrees are the prime factors of $n-1$. Applying to these abelian equations the method of No. 76, it follows by an easy induction that the equation (9) and consequently (No. 76), every abelian equation of prime degree is solvable by radicals.*

§16.—*Metacyclic or Galoisian Equations.*

78. In No. 74 we have represented the circular substitution $(x_0, x_1, \ldots x_{n-1})$ in the abbreviated form

$$\begin{pmatrix} x_z \\ x_{z+1} \end{pmatrix} \text{ or } |z\ z+1|.$$

Similarly any substitution

$$\begin{pmatrix} x_0 & x_1 & \ldots . & x_{n-1} \\ x_\alpha & x_\beta & \ldots . & x_\lambda \end{pmatrix}$$

($\alpha, \beta, \ldots . \lambda$ being some permutation of the numbers $0, 1, \ldots . n-1$) may be represented in the form

$$\begin{pmatrix} x_z \\ x_{\phi(z)} \end{pmatrix} \text{ or } |z\ \phi(z)|, \tag{1}$$

if $\phi(z)$ be a function of the index z which takes for $z = 0, 1 \ldots . n-1$ the values $\alpha, \beta, \ldots . \lambda$ respectively. It is easy to construct such a function, for instance by means of Lagrange's Interpolation-Formula.

If, in particular, n is a prime number, $n = p$, and we agree again to consider two indices which are congruent (mod p) as equivalent, then it is always possible to choose for $\phi(z)$ an integral function with integral coefficients; and conversely,

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*References concerning *uniserial Abelian* equations: Abel, Œuvres I; Jordan, No. 400, 401; Serret, No. 582, 588; Netto, §172. Further: Kronecker, Berliner Monatsberichte, 1858; Weber, Theorie der Abel'schen Zahlkörper, Acta Math., Vol. 8, 9.

Concerning the equation for the *division of the circle* in particular, see Bachman, Die Lehre von der Kreisteilung.

Concerning the most general kind of Abelian equations, so-called *manyfold* ("mehrfaltige," Kronecker) Abelian equations, see Abel, Œuvres I; Jordan, No. 402–415; Netto, §160–187; Serret, No. 529–554, and Kronecker, Berliner Monatsberichte, 1870.

any such function will be apt to represent a substitution between p letters if the p values which it takes for $z = 0, 1, \ldots . p-1$ form a complete system of remainders (mod n). Hermite has given, in this respect, the following criterion, necessary and sufficient:

Form the first $p-2$ powers of $\phi(z)$:

$$\phi(z), \ [\phi(z)]^2, \ldots . [\phi(z)]^{p-2},$$

and reduce them to the degree $p-1$ by means of Fermat's theorem:

$$z^p \equiv z \pmod{p},$$

then the coefficient of z^{p-1} in every one of these reduced functions must be divisible by p (see Serret, No. 474–476; Jordan, No. 114, 115; Netto, §§134, 135).

Remark: Since the order in which the letters in the first line of a given substitution are written may be chosen arbitrarily (No. 3), we may represent the substitution

$$\binom{x_z}{x_{\phi(z)}} \text{ also by the symbol } \binom{x_{\psi(z)}}{x_{\phi[\psi(z)]}},$$

$\psi(z)$ denoting a function satisfying Hermite's criterion ("Transformation of the index").

79. This "analytical representation" of a substitution is useful in solving the problem: *To determine the largest substitution-group between the letters* $x_0, x_1, \ldots . x_{p-1}$ *of which the cyclic group*

$$H = \binom{x_z}{x_{z+c}} \pmod{p}, \ c = 0, 1, \ldots . p-1$$

is a self-conjugate subgroup.

Let

$$g = \binom{x_z}{x_{\phi(z)}}$$

be any substitution of the sought group G; then the transformed of the substitution

$$h = \binom{x_z}{x_{z+1}}$$

by g must again belong to the self-conjugate subgroup H (No. 36), and consequently be some power of h, say

$$g^{-1}hg = h^a.$$

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Now we have (compare the remark in No. 78),

$$g^{-1} = \binom{x_{\phi(z)}}{x_z}, \; g^{-1}h = \binom{x_{\phi(z)}}{x_{z+1}}, \; g^{-1}hg = \binom{x_{\phi(z)}}{x_{\phi(z+1)}};$$

on the other hand, we may write

$$h^a = \binom{x_{\phi(z)}}{x_{\phi(z)+a}}.$$

Consequently $\psi(z)$ must satisfy the condition

$$\phi(z+1) \equiv \phi(z) + a \qquad (\text{mod } p),$$

for $z = 0, 1, 2, \ldots . Z \ldots . p-1$; that is,

$$\left.\begin{array}{ll} \phi(1) \equiv \phi(0) + a & \\ \phi(2) \equiv \phi(1) + a & \equiv \phi(0) + 2a, \\ \ldots\ldots\ldots\ldots & \ldots\ldots \\ \phi(Z) \equiv \phi(Z-1) + a & \equiv \phi(0) + Za, \end{array}\right\} (\text{mod } p),$$

or if we put $\phi(0) = b$ and write z instead of Z,

$$\phi(z) \equiv az + b \quad (\text{mod } p).$$

Therefore all the substitutions of the sought group G must be of the form

$$\binom{x_z}{x_{az+b}}.$$

Conversely, it is easily seen that this symbol always represents a substitution, provided a be not divisible by p. There are on the whole $p(p-1)$ *different* substitutions of this form, corresponding to the values

$$a \equiv 1, 2, \ldots . p-1; \; b \equiv 0, 1, \ldots . p-1 \qquad (\text{mod } p).$$

These $p(p-1)$ substitutions form a group called the *metacyclic group;* this is then the largest group between the letters $x_0, x_1, \ldots . x_{p-1}$ that contains the given cyclic group as a self-conjugate subgroup.

Besides the powers of the substitution $\binom{x_z}{x_{z+1}}$, the metacyclic group contains no other circular substitution of the order p. For the substitution

$$\binom{x_z}{x_{az+b}}$$

leaves, if a is not congruent 1 (mod p), one root unaltered, viz. that one whose index is determined by the congruence

$$az + b \equiv z \pmod{p}.$$

80. An equation of the p^{th} degree,

$$f(x) = 0, \tag{1}$$

whose group G is the metacyclic group, is called a *metacyclic equation*. It has the following two characteristic properties:

a). It is *irreducible*, for the metacyclic group is evidently transitive (No. 65).

b). *All its roots are rationally expressible in terms of two of them.*

For it is easily seen that the substitution 1 is the only substitution of the metacyclic group which leaves at the same time the two roots x_0 and x_1 unaltered. Hence the function $V = m_0 x_1 + m_1 x_1$ "belongs" (No. 67) to the group 1, and therefore all the roots are rationally expressible in terms of V, or what amounts to the same, in terms of x_0 and x_1:

$$x_2 = \theta_2(x_0, x_1); \; \ldots \; x_{p-1} = \theta_{p-1}(x_0, x_1). \tag{2}$$

Now an *irreducible* equation of *prime degree, whose roots are rationally expressible in terms of two of them*, is called a *Galoisian equation*. Thus we have the result:

A metacyclic equation is always a Galoisian equation.

81. Let us now, conversely, determine the group of a Galoisian equation of the p^{th} degree.

a). Since the equation is supposed to be irreducible, its group G must be *transitive*, and therefore its *order divisible by the degree p* (No. 24).

b). Since, moreover, p is supposed to be prime, G *must contain a cyclic subgroup H, of order p*, according to Cauchy's theorem (No. 22); we may suppose that it consists of the different powers of the circular substitution

$$a = (x_0, x_1, \ldots \ldots x_{p-1}),$$

x_0 and x_1 denoting the two roots in terms of which all the others are supposed to be expressible rationally.*

c). This cyclic subgroup H must be *self-conjugate* under G. In fact, if it

* For among the different powers of any circular substitution of the order p there is always one which replaces x_0 by x_1.

were otherwise, G would contain two circular substitutions of the order p which are not powers of one another (No. 36), but this is impossible.

For let $b = (x_{i_0}, x_{i_1}, \ldots x_{i_{p-1}})$ be another circular substitution* of the order p contained in G. It is then always possible to choose two integers, μ and ν, *incongruent* (mod p), such that

$$i_{\mu+1} - i_\mu \equiv i_{\nu+1} - i_\nu, \text{ say } \equiv k \pmod{p}.$$

Applying, then, according to No. 62, B′, the two substitutions $b^\mu a^{-i_\mu}$ and $b^\nu a^{-i_\nu}$ of G to the rational relations

$$x_{i_\alpha} = \theta_\alpha(x_{i_0}, x_{i_1}),$$
$$(\alpha = 0, 1, \ldots p-1)$$

we obtain

$$x_{i_{\alpha+\mu}-i_\mu} = \theta_\alpha(x_0, x_\kappa),$$
$$x_{i_{\alpha+\nu}-i_\nu} = \theta_\alpha(x_0, x_\kappa),$$

and consequently

$$i_{\alpha+\mu} - i_\mu = i_{\alpha+\nu} - i_\nu \qquad (\alpha = 0, 1, \ldots p-1).$$

Hence follows† that the p differences $i_{\beta+1} - i_\beta$ $(\beta = 0, 1, \ldots p-1)$ have the same value, say m; that is to say, $b = a^m$. Q. E. D.

d). Since the cyclic group H is a self-conjugate subgroup of G, G must be the metacyclic group or one of its transitive subgroups (No. 79), thus we have the result:

The group of a Galoisian equation is the metacyclic group or one of its transitive subgroups.

82. In order to solve a metacyclic equation we construct (No. 58) a rational function ψ of the roots which belongs to the cyclic subgroup H; ψ satisfies, then, (No. 67) a resolvent equation of the degree $p-1$: $g(\psi) = 0$. To determine its group Γ (No. 71) we remark that the metacyclic group may be generated by the two substitutions

$$a = \begin{pmatrix} x_s \\ x_{s+1} \end{pmatrix} \text{ and } c = \begin{pmatrix} x_s \\ x_{gs} \end{pmatrix},$$

g denoting a primitive root of the prime number p. The substitutions of the metacyclic group are then $a^\alpha c^\gamma$ $(\alpha = 0, 1, \ldots p-1;\ \gamma = 1, 2, \ldots p-1)$, and

* We may suppose $i_0 = 0$, $i_1 = 1$ by replacing, eventually, b by a convenient power of b.

† See Kronecker, Berliner Monatsberichte, 1879.

those of the cyclic subgroup H are a^α $(\alpha = 0, 1, \ldots p-1)$. Hence the $p-1$ conjugate values of ψ under G are (No. 23)

$$\psi_0 = \psi, \ \psi_1 = \psi_c, \ \psi_2 = \psi_{c^2}, \ \ldots \ \psi_{p-2} = \psi_{c^{p-2}}.$$

The cyclic group H being self-conjugate under G, the ψ's undergo the substitution $(\psi_0, \psi_1, \ldots \psi_{p-2})^\beta$, when the x's are operated upon by the substitution $a^\alpha b^\beta$, and therefore (No. 71) the group Γ of the resolvent equation $g(\psi) = 0$ is the cyclic group of order $p-1$, consisting of the different powers of the substitution $(\psi_0, \psi_1, \ldots \psi_{p-2})$, and consequently the resolvent is an Abelian equation.

By the adjunction of ψ the group of the equation $f(x) = 0$ is reduced to the cyclic subgroup H (No. 70); the equation $f(x) = 0$ is therefore an Abelian equation in the new domain of rationality.

Analogous results are obtained for the transitive subgroups of the metacyclic group, which are all contained in the form

$$a^\alpha c^{\delta\gamma},$$
$$\left(\alpha = 0, 1, \ldots p-1; \ \gamma = 1, 2, \ldots \frac{p-1}{\delta}\right),$$

δ denoting any divisor of $p-1$. Thus we find the result:

A Galoisian equation can always be solved by a chain of two Abelian equations, the first of degree $p-1$ $\left(\text{resp. } \frac{p-1}{\delta}\right)$, *the second of degree* p, and consequently (No. 75):

A Galoisian equation is always solvable by a chain of Abelian equations of prime degree; and finally (No. 76):

A Galoisian equation is always solvable by radicals.

83. An important example of Galoisian equations are the *binomial equations of prime degree:*

$$x^p = A,$$

where A is supposed to be a rational quantity which is not expressible as a p^{th} power of another rational quantity.

For, in the first place, the binomial equation is *irreducible* under these circumstances, as Abel has proved (see Jordan, No. 418; Netto, §203).

Further, if x_0 be one of its roots and ω an imaginary p^{th} root of unity, the other roots are

$$x_1 = \omega x_0, \ x_2 = \omega^2 x_0, \ \ldots \ x_{p-1} = \omega^{p-1} x_0,$$

hence $\omega = \frac{x_1}{x_0}$ and consequently

$$x_\alpha = \left(\frac{x_1}{x_0}\right)^\alpha x_0 \qquad (\alpha = 0, 1, \ldots . p-1).$$

All the roots are therefore rationally expressible in terms of two of them. The equation is therefore indeed a Galoisian equation.

For the function ψ of No. 82 which belongs to the cyclic subgroup, we may choose, in this case, the quantity ω, since

$$\omega = \frac{x_1}{x_0} = \frac{x_2}{x_1} = \ldots . = \frac{x_0}{x_{p-1}}.$$

ω satisfies the resolvent equation of the degree $p-1$:

$$\omega^{p-1} + \omega^{p-2} + \ldots . + \omega + 1 = 0,$$

which is, in fact, an Abelian* equation (No. 77).

After the adjunction of ω the binomial equation is itself an Abelian, the function $\theta(x_0)$ being in this case $\theta(x_0) = \omega x_0$ (compare also No. 64).

Hence follows that *a binomial equation of prime degree is always solvable by a chain of Abelian equations of prime degree.*†

§17.—*Solution by Radicals.*

84. The equation of the n^{th} degree

$$f(x) = 0 \tag{1}$$

is said to be *solvable by radicals* if its roots can be derived from the known quantities $\Re', \Re'', \ldots .$ by a finite number of extractions of roots whose exponents may, without loss of generality, be supposed to be prime. If $\xi, \eta, \ldots . \psi$ denote all the radicals which occur in the expressions for all the roots $x_1, x_2, \ldots . x_n$, the solution may be written in the form of a chain of binomial equations of prime degree:

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* For certain special domains of rationality it may, however, happen that ω satisfies an Abelian of degree $\frac{p-1}{\delta}$.

† This holds still if the binomial equation is reducible; for, in this case, one root is rational (Netto, §208), and the problem is reduced to the determination of ω.

References concerning Galoisian equations: Jordan, No. 416–418; Netto, §187–190; Kronecker, Berliner Monatsberichte, 1879.

$$\left.\begin{array}{l} \xi^\lambda = L(\mathfrak{R}', \mathfrak{R}'', \ldots\ldots), \\ \eta^\mu = M(\xi;\ \mathfrak{R}', \mathfrak{R}'', \ldots\ldots), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \psi^\rho = P(\phi, \ldots\ldots \eta, \xi;\ \mathfrak{R}', \mathfrak{R}'', \ldots\ldots), \\ x_\alpha = R_\alpha(\psi, \phi, \ldots\ldots \eta, \xi;\ \mathfrak{R}', \mathfrak{R}'', \ldots\ldots), \\ \alpha = 1, 2, \ldots\ldots n, \end{array}\right\} \qquad (2)$$

$L, M, \ldots\ldots P, R_\alpha$ denoting rational functions with integral coefficients,* and the degrees $\lambda, \mu, \ldots\ldots \rho$ being prime numbers.

Now, according to No. 83, every one of these binomial equations, and consequently also the whole chain, can be replaced by a chain of Abelian equations of prime degrees, and since conversely every Abelian equation is solvable by radicals, we have the result:

In order that an equation be solvable by radicals, it is necessary and sufficient that it be reducible to a chain of Abelian equations of prime degrees:

$$\left.\begin{array}{l} \Phi(y;\ \mathfrak{R}', \mathfrak{R}'', \ldots\ldots) = 0, \\ \Psi(z;\ y, \mathfrak{R}', \mathfrak{R}'', \ldots\ldots) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Lambda(w;\ v, \ldots\ldots z, y, \mathfrak{R}', \mathfrak{R}'', \ldots\ldots) = 0, \\ x_\alpha = R_\alpha(w, v, \ldots\ldots y, \mathfrak{R}', \mathfrak{R}'', \ldots\ldots), \\ \alpha = 1, 2, \ldots\ldots n. \end{array}\right\} \qquad (3)$$

The first of these equations is an Abelian equation with respect to the domain $(\mathfrak{R}', \mathfrak{R}'', \ldots\ldots)$, the second with respect to the enlarged domain $(y, \mathfrak{R}', \mathfrak{R}'', \ldots\ldots)$ and so forth.

Since the group of a binomial equation is more complicated than that of an Abelian (No. 83), we have, from our present point of view, to consider as the simplest possible class of equations not the binomial, but the Abelian equations of prime degree, and therefore it is preferable to start, in our developments, from the chain of Abelian equations (3).

85. Let now G denote the group of the equation (1)—which we suppose to be solvable by the chain (3)—with respect to the original domain of rationality $(\mathfrak{R}', \mathfrak{R}'', \ldots\ldots)$.

We begin the solution by solving the first equation of the chain (3) and adjoining one of its roots, y, to our domain; the group G of (1) is then

*It is always to be understood that some of the quantities $\xi, \eta, \ldots\ldots \psi$ may be wanting in the expression of R_α, and similarly for the other functions.

reduced to a certain subgroup (No. 69), say H (including the case $H = G$). Next we solve the second equation and adjoin one of its roots, z, to our present domain; the group H of (1) is then reduced to a subgroup, say I (including again the case $I = H$). Continuing in this way, we arrive at last, after solving the last equation, at the domain

$$(w, v, \ldots . z, y, \mathfrak{R}', \mathfrak{R}'', \ldots .);$$

with respect to this domain the group of (1) *is* 1, since all the roots are rationally known with respect to it (No. 63, a).

Now I say: By every one of these successive adjunctions the group of (1) is either not reduced at all or it is reduced to a *self-conjugate* subgroup of *prime index.*

Proof: I). Let ν be the index of the subgroup H, to which the group G of (1) is reduced after the adjunction of the root $y_1 = y$ of the first Abelian equation

$$\Phi(y;\ \mathfrak{R}', \mathfrak{R}'', \ldots .) = 0, \tag{4}$$

whose degree, p, is supposed to be prime.

a). Let, moreover, $\phi(x_1, x_2, \ldots . x_n)$ denote a rational function of the roots which belongs to the group H (No. 67 and 58); ϕ will then satisfy a resolvent equation

$$g(\phi) = 0, \tag{5}$$

of degree ν and irreducible in the domain $(\mathfrak{R}', \mathfrak{R}'', \ldots .)$ (No. 71).

b). On the other hand, H being the group of (1) in the enlarged domain

$$(y_1, \mathfrak{R}', \mathfrak{R}'', \ldots .),$$

ϕ is rationally expressible in terms of y_1 (No. 62, A):

$$\phi = r(y_1). \tag{6}$$

Applying now the theorem of No. 71, 5 to the rational function $\phi = r(y_1)$ of the root y_1 of the irreducible (since Abelian) equation (4), we obtain a second resolvent equation for ϕ:

$$h(\phi) = 0, \tag{7}$$

likewise irreducible and whose degree is a divisor of the degree p of (6) and consequently either $= 1$ or $= p$, since p is prime.

c). But the two equations (5) and (7), being both irreducible and having the root ϕ in common, must be identical (No. 57), and therefore the degree ν of (5), that is to say, the index of H under G, must be either $= 1$ or $= p$. Thus we

find: *By the adjunction of a root $y_1 = y$ of an irreducible equation of prime degree p, the group of* (1) *is either not reduced at all or it is reduced to a subgroup of index p.*

II. In the latter case the p roots of (7), which are

$$r(y_1), \; r(y_2), \; \ldots . \; r(y_p),$$

$y_1, y_2, \ldots . y_p$ denoting the p roots of (4), must be identical with the roots of (5), which are the $\nu = p$ conjugate values $\phi_1 = \phi, \phi_2 \ldots . \phi_p$ of ϕ under G, derived from ϕ by certain substitutions $g_1 = 1, g_2, \ldots . g_p$ of G. We may therefore write

$$\phi_\alpha = r(y_\alpha), \tag{8}$$

$$\alpha = 1, 2, \ldots . p.$$

Now the function ϕ_α of the roots of (1) belongs to the subgroup $g_\alpha^{-1} H g_\alpha$ of G (No. 35), and any rational function of the roots which remains unaltered by the same group $g_\alpha^{-1} H g_\alpha$ is rationally expressible in terms of ϕ_α (No. 67) and consequently also in terms of y_α.

If, therefore, instead of the root $y_1 = y$ of (4), we adjoin the root y_α, the group G' of (1) with respect to the domain $(y_\alpha, \Re', \Re'', \ldots .)$ must be either $g_\alpha^{-1} H g_\alpha$ or one of its subgroups (No. 63, a); but the latter case is impossible; for, according to No. 85, the index of G' under G must be either 1 or p, and the index of $g_\alpha^{-1} H g_\alpha$ under G is p, hence $G' = g_\alpha^{-1} H g_\alpha$. Thus we find:

If, instead of the root y_1, we adjoin another root y_α of (4), *the group G is reduced to a subgroup $H_\alpha = g_\alpha^{-1} H g_\alpha$* CONJUGATE WITH H UNDER G.

III. If now, in particular, the auxiliary equation (4) is an *Abelian* equation, as we suppose, then y_α is rationally expressible in terms of y_1:

$$y_\alpha = \theta(y_1);$$

but also, conversely, y_1 is rationally expressible in terms of y_α; for, according to No. 73, we have

$$\theta^p(y_1) = y_1,$$

therefore

$$\theta^{p-1}(y_\alpha) = \theta^p(y_1) = y_1.$$

Hence the two domains of rationality $(y_1, \Re', \Re'', \ldots .)$ and $(y_\alpha, \Re', \Re'', \ldots .)$ are identical and the group of (1) must be the same with respect to either of them. Therefore $g_\alpha^{-1} H g_\alpha = H$, for $\alpha = 1, 2, \ldots . p$, consequently (No. 36) H is a self-conjugate subgroup of G. Thus we have the result:

By the adjunction of a root of an Abelian equation of prime degree p, the group

of the given equation is either not reduced at all, or it is reduced to a self-conjugate subgroup of index p.

Remark: To appreciate fully the importance of this theorem, which is due to Galois, we must bear in mind that *the adjoined quantity y is not supposed to be a rational function of the roots $x_1, x_2, \ldots x_n$*; hence this theorem comprises, as will be seen presently, Abel's celebrated theorem that the radicals which enter into the solution of a solvable equation are always rationally expressible in terms of the roots and of certain roots of unity.

86. Applying this theorem to every one of the successive Abelian equations of the chain (3), we find that the different groups through which we pass in the process of successive adjunction must form a series of groups beginning with G, ending with 1, and such that each group is a self-conjugate subgroup of prime index of the group immediately preceding, or, according to No. 43: The factors of composition of the group G must all be prime numbers.

But this condition is also sufficient. For if $\lambda, \mu, \ldots \rho$ denote the factors of composition of the group G, then we can, according to No. 72, solve the given equation by a chain of simple regular equations of the degrees $\lambda, \mu, \ldots \rho$; and if, in particular, the factors of composition are all prime numbers, then these regular equations are Abelian equations of prime degrees (No. 71, 4), and therefore the given equation is solvable by radicals.

Thus we have proved Galois' criterion for the solution by radicals: *In order that an equation be solvable by radicals, it is necessary and sufficient that the factors of composition of its group be all prime numbers.*

Remark: At the same time we see from No. 72 that we may choose, for the auxiliary quantities $y, z, \ldots v, w$ of the chain (3), *rational functions of the roots $x_1, x_2, \ldots x_n$.* If, then, we transform the chain of Abelian equations into a chain of binomial equations, by means of the theorem of No. 76, we obtain Abel's theorem mentioned in the remark in No. 85.*

87. The group of the general equation of the n^{th} degree is the symmetric group (No. 66), whose factors of composition are for $n > 4$ (No. 44): 2 and $\frac{n!}{2}$; the latter being composite, *the general equation of a higher than the fourth degree is not solvable by radicals.*

* Since the above was written, an important paper on the solution of equations by means of a chain of auxiliary equations has been published by Hölder: Zurückführung einer beliebigen algebraischen Gleichung auf eine Kette von Gleichungen (Math. Annalen, Bd. 34).

For $n=3$ the factors are $2, 3$; for $n=4$ they are $2, 3, 2; 2$ (No. 43), all of them prime; this is the reason why the general equations of the third and fourth degree are solvable by radicals.

Their solution by a chain of Abelian equations is exhibited in the following tables:

a). *Cubic equation* (Notations of No. 14):

$$x^3 - c_1x^2 + c_2x - c_3 = 0. \tag{1}$$

Domain of rationality.	Group of (1)
Original domain: (c_1, c_2, c_3): $\phi = (x_2 - x_3)(x_3 - x_1)(x_1 - x_2)$ belongs to G_3; $\phi^2 - \Delta = 0$, Abelian.	G_6
Adjoin ϕ: x_1 belongs to $G_1 = 1$; and (1) itself is Abelian.	G_3
Adjoin x_1: The other roots are now rationally known; their expressions are given in Serret, No. 511.	1

b). *Biquadratic equation*, Ferrari's solutiòn (Notations of No. 15–18):

$$x^4 - c_1x^3 + c_2x^2 - c_3x + c_4 = 0. \tag{1}$$

Domain of rationality.	Group of (1)
Original domain (c_1, c_2, c_3, c_4): $\phi = \Pi(x_\alpha - x_\beta)$ belongs to G_{12}; $\phi^2 - \Delta = 0$, Abelian.	G_{24}
Adjoin ϕ: $\xi_1 = x_1x_2 + x_3x_4$ belongs to G_4, $\xi^3 - c_2\xi^2 + (c_1c_3 - 4c_4)\xi - [c_4(c_1^2 - 4c_2) + c_3^2] = 0$, Abelian.	G_{12}

Adjoin ξ_1: $\eta = x_1 + x_2 - x_3 - x_4$ belongs to G_2, $\eta^2 - 4\xi_1 - c_1^2 + 4c_2 = 0$, Abelian.	G_4
Adjoin η: x_1 belongs to $G_1 = 1$, $x^2 - \left(\frac{c_1+\eta}{2}\right)x + \left(\frac{\xi_1}{2} + \frac{c_1\xi_1 - 2c_3}{2\eta}\right) = 0$, Abelian.	G_2
Adjoin x_1: The other roots are now rationally known; this is evident for x_2; to find the expressions for x_3 and x_4, notice that $\xi_2 - \xi_3 = (x_1 - x_2)(x_3 - x_4)$ is rationally expressible in terms of ξ_1 and $\sqrt{\Delta}$ (see Serret, No. 511, equation (3)).	1

88. Let us now determine all *irreducible* equations of *prime degree* which are *solvable by radicals*.

Let
$$f(x) = 0 \tag{1}$$
be the given equation of prime degree p, G its group and
$$G - H - \ldots . K - L \ldots . 1, \tag{2}$$
a series of composition of G.

The factors of composition are then all prime numbers since (1) is supposed to be solvable by radicals (No. 86).

I). The group G must be transitive, since (1) is supposed to be irreducible (No. 65), and therefore its order must be divisible by p (No. 24); since, moreover, p is supposed to be prime, G *must contain a circular substitution of order* p (No. 22), say
$$t = \begin{pmatrix} x_z \\ x_{z+1} \end{pmatrix} \pmod{p}.$$

In the series (2) there must necessarily exist one group, K, such that this substitution t is contained in K and in all preceding groups, but is not contained in the group L immediately following K.

I say, *this group L must be the group* 1.

Proof: a). Let ν be the index of L under K; ν being prime, we may apply to the substitution t the lemma of No. 40: "If k be any substitution of K which is not contained in L, then k^ν, and no lower power of k, belongs to K; moreover, the order of the substitution k is a multiple of ν."

Now evidently $t^p = 1$ is the lowest power of t which belongs to L, therefore $\nu = p$.

b). Suppose, now, L contains a substitution s different from 1, which replaces the letter x_α by a different letter x_β. The substitution $u = st^{\alpha - \beta}$ will then leave the letter x_α unaltered.

Now u belongs to the group K, but not to L, since $\alpha - \beta$ is not divisible by p according to our assumptions; therefore, according to the lemma of No. 40, the order of u must be divisible by p, but this is impossible since u is a substitution between p letters, which leaves the letter x_α unaltered.

Consequently L can contain no substitution besides 1: $L = 1$.

II. Let us now build up the series of composition, starting from the last group $L = 1$.

a). The index of L under K is p according to I, a; therefore K *must be the cyclic group* of order p consisting of the different powers of the circular substitution t, since K is supposed to contain t.

b). Hence the group immediately preceding K in the series (2), say J, must be contained in the metacyclic group, since it contains the cyclic group of order p, viz. K, as a self-conjugate subgroup (No. 79). According to the remark at the end of No. 79, J contains then no circular substitution of order p except the powers of the substitution t.

c). Let now I be the group immediately preceding J in the series (2); J being a self-conjugate subgroup of I, the transformed of t by any substitution of I, which is again a circular substitution of order p, must belong to J (No. 36) and must therefore be some power of t, according to *b*). Hence follows that the cyclic group K must be self-conjugate not only under J but also under I, and therefore I must be contained in the metacyclic group.

d). The same conclusion holds for the group immediately preceding I, and so forth, and finally for the group G itself: G must be the metacyclic group or one of its transitive subgroups.

According to No. 81, we have therefore the theorem, due to Galois:

Every irreducible equation of prime degree which is solvable by radicals is a Galoisian equation.

The conversion has been proved previously in No. 82.*

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* References concerning §17: Serret, No. 580 and No. 587–598; Jordan, No. 373–378, 520; Kronecker, Berliner Monatsberichte, 1879.

Concerning solvable equations whose degree is not prime, see Jordan, Livre IV; besides Journal de Mathématiques, Ser. 2, Vol. 12, 13, 14, Netto, §233–243.

As to applications of Galois' theory to the *division of trigonometric, the division and transformation of elliptic and Abelian functions*, see Jordan, Chap. IV; Weber, Zur Theorie der elliptischen Functionen, Acta Math., Vol. 6; Kronecker, Entwicklungen aus der Theorie der Gleichungen, Berliner Monatsb. 1879.

As to applications to geometrical problems, see Jordan, Chap. III.

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BALTIMORE, MD., May 1889.

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* To the text-books enumerated at the beginning of this paper may be added *Chrystal's Algebra*, whose second volume (published since the above was written) contains a chapter on the theory of substitutions.

Quelques propriétés des nombres K_n^p.

PAR M. M. D'OCAGNE.

1. Les nombres que j'ai ici en vue sont ceux que j'ai étudiés dans un Mémoire paru en 1887 dans l'*American Journal of Mathematics* (p. 353), où je les définissais au moyen d'un triangle arithmétique analogue à celui de Pascal. Les renvois que l'on trouvera ici se rapportent à ce Mémoire.

2. Soit un polynôme entier en x

$$F(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n.$$

Ce polynôme peut toujours être mis sous la forme

$$F(x) = b_0 + b_1 x + b_2 x(x-1) + \ldots\ldots + b_n x(x-1)\ldots\ldots(x-(n-1)).$$

Cherchons à calculer les coefficients b au moyen des coefficients a. Nous avons en donnant à x des différences successives égales à 1,

$$\begin{aligned}\Delta^p F(x) = 1.2\ldots\ldots p.b_p &+ 2.3\ldots\ldots(p+1)\,b_{p+1}x \\ &+ 3.4\ldots\ldots(p+2)\,x(x-1) + \ldots\ldots\end{aligned}$$

Donc $$\Delta^p F(0) = 1.2\ldots\ldots p.b_p$$

et $$b_p = \frac{\Delta^p F(0)}{1.2\ldots\ldots p}.$$

Mais la formule (15) de mon Mémoire donne

$$\Delta^p F(x) = \sum_{i=0}^{i=n-p} \frac{K_{p+i}^p}{(p+i)(p+i-1)\ldots\ldots(p+1)} F^{p+i}(x).$$

Or, $$F^{p+i}(0) = (p+i)(p+i-1)\ldots\ldots(p+1)p\ldots\ldots 2.1\,a_{p+i};$$

par suite

$$\Delta^p F(0) = 1.2\ldots\ldots p \sum_{i=0}^{i=n-p} K_{p+i}^p a_{p+i},$$

et $$b_p = K_p^p a_p + K_{p+1}^p a_{p+1} + K_{p+2}^p a_{p+2} + \ldots\ldots + K_n^p a_n. \qquad (\alpha)$$

C'est de cette curieuse formule que nous allons ici tirer quelques propriétés des nombres K_m^p.

3. En premier lieu, faisons

$$F(x) = 1 + x^n.$$

La formule (α) donne alors

$$b_p = K_n^p.$$

Par suite

$$1 + x^n = 1 + K_n^1 x + K_n^2 x(x-1) + \dots + K_n^n x(x-1) \dots (x-(n-1))$$

ou

$$x^{n-1} = K_n^1 + K_n^2 (x-1) + \dots + K_n^n (x-1)(x-2) \dots (x-(n-1)). \quad (\beta)$$

Nous retrouvons ainsi par une tout autre voie que celle contenue dans notre Mémoire la formule (24) de celui-ci.

4. Faisons maintenant

$$F(x) = (1+x)^n = 1 + C_n^1 x + C_n^2 x^2 + \dots + C_n^n x^n.$$

Ici la formule (α) donne

$$b_p = K_p^p C_n^p + K_{p+1}^p C_n^{p+1} + \dots + K_n^p C_n^n.$$

Mais, d'après la formule (β) où on remplace x par $x+1$ et n par $n+1$, on a

$$b_p = K_{n+1}^{p+1};$$

conséquemment

$$K_{n+1}^{p+1} = K_p^p C_n^p + K_{p+1}^p C_n^{p+1} + \dots + K_n^p C_n^n,$$

formule qui peut encore s'écrire

$$K_n^p = K_{n-1}^{p-1} + C_{n-1}^1 K_{n-2}^{p-1} + C_{n-1}^2 K_{n-3}^{p-1} + \dots + C_{n-1}^{n-p} K_{p-1}^{p-1}. \quad (\gamma)$$

Il est alors curieux de la comparer à la formule (3) de notre Mémoire. En effet, chacune de ces formules donne K_n^p en fonction linéaire de

$$K_{n-1}^{p-1},\ K_{n-2}^{p-1},\ \dots,\ K_{p-1}^{p-1},$$

c'est-à-dire, en nous reportant au triangle arithmétique de définition, de tous les nombres situés au-dessus de la rangée horizontale de K_n^p, dans la colonne qui précède immédiatement la sienne. Or, les coefficients de cette fonction linéaire sont, dans la formule (γ),

$$1,\ C_{n-1}^1,\ C_{n-1}^2,\ \dots,\ C_{n-1}^{n-p},$$

et dans la formule (3),

$$1,\ p,\ p^2,\ \dots,\ p^{n-p}.$$

On a, par exemple,

$$K_5^3 = K_4^2 + C_4^1 K_3^2 + C_4^2 K_2^2 = 7 + \frac{4}{1}.3 + \frac{4.3}{1.2}.1 = 25$$

et
$$K_5^3 = K_4^2 + 3.K_3^2 + 3^2.K_2^2 = 7 + 3.3 + 9.1 = 25.$$

En convenant, comme nous l'avons fait dans notre Mémoire, de prendre $K_m^p = 0$ lorsque $p > m$, on peut écrire symboliquement la formule (γ), d'après la notation que nous avons proposée dans le Bulletin de la Société Mathématique de France (T. XV, p. 156),

$$K_n^p = (K^{p-1} + 1)^{n-1}. \qquad (\gamma')$$

La formule (γ) peut se vérifier *à posteriori* de la manière suivante :

Il suffit de faire voir que la formule supposée vraie pour les nombres de la rangée horizontale d'indice $m - 1$ l'est encore pour la rangée d'indice m.

Admettons donc que l'on ait

$$K_{n-1}^p = K_{n-2}^{p-1} + C_{n-2}^1 K_{n-3}^{p-1} + \ldots\ldots + C_{n-2}^{n-p-1} K_{p-1}^{p-1},$$
$$K_{n-1}^{p-1} = K_{n-2}^{p-2} + C_{n-2}^1 K_{n-3}^{p-2} + \ldots\ldots + C_{n-2}^{n-p-1} K_{p-1}^{p-2} + C_{n-2}^{n-p} K_{p-2}^{p-2}.$$

Alors

$$pK_{n-1}^p + K_{n-1}^{p-1} = pK_{n-2}^{p-1} + K_{n-2}^{p-2} + C_{n-2}^1 (pK_{n-3}^{p-1} + K_{n-3}^{p-2}) + \ldots\ldots$$
$$+ C_{n-2}^{n-p-1} (pK_{p-1}^{p-1} + K_{p-1}^{p-2}) + C_{n-2}^{n-p} K_{p-2}^{p-2}.$$

Mais la formule (1) de notre Mémoire donne

$$pK_{n-1}^p + K_{n-1}^{p-1} = K_n^p,$$
$$pK_{n-i}^{p-1} + K_{n-i}^{p-2} = (p-1) K_{n-i}^{p-1} + K_{n-i}^{p-2} + K_{n-i}^{p-1} = K_{n-i+1}^{p-1} + K_{n-i}^{p-1}.$$

L'égalité précédente devient donc

$$K_n^p = K_{n-1}^{p-1} + K_{n-2}^{p-1} + C_{n-2}^1 (K_{n-2}^{p-1} + K_{n-3}^{p-1}) + \ldots\ldots$$
$$+ C_{n-2}^{n-p-1} (K_p^{p-1} + K_{p-1}^{p-1}) + C_{n-2}^{n-p} K_{p-1}^{p-1}$$
$$= K_{n-1}^{p-1} + (1 + C_{n-2}^1) K_{n-2}^{p-1} + \ldots\ldots$$
$$+ (C_{n-2}^{n-p-2} + C_{n-2}^{n-p-1}) K_p^{p-1} + (C_{n-2}^{n-p-1} + C_{n-2}^{n-p}) K_{p-1}^{p-1}$$
$$= K_{n-1}^{p-1} + C_{n-1}^1 K_{n-2}^{p-1} + \ldots\ldots + C_{n-1}^{n-p-1} K_p^{p-1} + C_{n-1}^{n-p} K_{p-1}^{p-1};$$

et la vérification se trouve ainsi faite.

5. Faisons enfin

$$F(x) = (x-1)(x-2) \ldots\ldots (x-n),$$

ou, en représentant, comme dans notre Mémoire, par S_m^n la somme des produits n à n des m premiers nombres,

$$F(x) = (-1)^n S_n^n + (-1)^{n-1} S_n^{n-1} x + (-1)^{n-2} S_n^{n-2} x^2 + \ldots\ldots - S_n^1 x^{n-1} + x^n.$$

La formule (α) donne ici

$$b_p = (-1)^{n-p} K_p^p S_n^{n-p} + (-1)^{n-p-1} K_{p+1}^p S_n^{n-p-1} + \ldots . - K_{n-1}^p S_n^1 + K_n^p. \quad (\delta)$$

Mais on a identiquement

$$\begin{aligned} F(x) &= x(x-1)\ldots.(x-(n-1)) \\ &\quad - n(x-1)(x-2)\ldots.(x-(n-1)), \\ (x-1)(x-2)\ldots.(x-(n-1)) &= x(x-1)\ldots.(x-(n-2)) \\ &\quad -(n-1)(x-1)(x-2)\ldots.(x-(n-3)), \\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ (x-2)(x-1) &= x(x-1) - 2(x-1), \\ x-1 &= x-1. \end{aligned}$$

Multipliant respectivement ces identités par

$$1,\ -n,\ n(n-1), \ldots.,\ (-1)^{n-2} n(n-1)\ldots. 3,\ (-1)^{n-1} n(n-1)\ldots. 3.2,$$

et faisant la somme, on a

$$\begin{aligned} F(x) &= (-1)^n n(n-1)\ldots. 2.1 + (-1)^{n-1} n(n-1)\ldots. 2.x \\ &\quad + (-1)^{n-2} n(n-1)\ldots 3.x(x-1) + \ldots + x(x-1)\ldots(x-(n-1)). \end{aligned}$$

Par conséquent,

$$b_p = (-1)^{n-p} n(n-1)\ldots.(p+1),$$

et la formule (δ) devient

$$\begin{aligned} K_n^p - S_n^1 K_{n-1}^p + S_n^2 K_{n-2}^p - \ldots. + (-1)^{n-p-1} S_n^{n-p-1} K_{p+1}^p + (-1)^{n-p} S_n^{n-p} K_p^p \\ = (-1)^{n-p} n(n-1)\ldots.(p+1). \quad (\varepsilon) \end{aligned}$$

Nous allons de cette formule en déduire une autre plus simple. Pour cela, remarquons que l'on a de même

$$\begin{aligned} K_n^{p-1} - S_n^1 K_{n-1}^{p-1} + S_n^2 K_{n-2}^{p-1} - \ldots. + (-1)^{n-p} S_n^{n-p} K_p^{p-1} \\ + (-1)^{n-p+1} S_n^{n-p+1} K_{p-1}^{p-1} = (-1)^{n-p+1} n(n-1)\ldots. p. \end{aligned}$$

Multiplions l'égalité (ε) par p et ajoutons là à la dernière, en remarquant que

$$p K_{n-i}^p + K_{n-i}^{p-1} = K_{n-i+1}^p.$$

Il vient

$$K_{n+1}^p - S_n^1 K_n^p + S_n^2 K_{n-1}^p - \ldots + (-1)^{n-p} S_n^{n-p} K_{p+1}^p + (-1)^{n-p+1} S_n^{n-p+1} K_p^p = 0. \ (\zeta)$$

Ici se place une curieuse remarque. Il semble au premier abord que la propriété exprimée par cette égalité ne diffère pas de celle qui se traduit par la

formule (11) de notre Mémoire, que nous reproduisons ici

$$K_{m+p}^p - S_p^1 K_{m+p-1}^p + S_p^2 K_{m+p-2}^p - \ldots + (-1)^p S_p^p K_m^p = 0.$$

En effet, le premier membre de chacune de ces égalités est de la forme

$$K_\mu^\nu - S_\lambda^1 K_{\mu-1}^\nu + S_\lambda^2 K_{\mu-2}^\nu - S_\lambda^3 K_{\mu-3}^\nu + \ldots,$$

ce développement étant prolongé jusqu'à ce qu'il s'arrête de lui-même, c'est-à-dire jusqu'à ce qu'on arrive à un terme contenant soit S_λ^λ, soit K_ν^ν.

Cette expression donne le premier membre de la formule (ζ) lorsqu'on y fait $\lambda = \mu - 1$, et le premier membre de la formule (11) lorsqu'on y fait $\lambda = \nu$.

Or, on peut vérifier que cette expression ne s'annule pas toujours, quels que soient λ, μ, ν. Prenons, par exemple, $\lambda = 2$, $\mu = 5$, $\nu = 3$. Il vient

$$K_5^3 - S_2^1 K_4^3 + S_2^2 K_3^3 = 25 - 3.6 + 2.1 = 9.$$

Il suit de là que, malgré l'analogie frappante de leur forme, les formules (ζ) et (11) expriment des propriétés parfaitement distinctes des nombres K_m^p.

6. Cette différence s'accuse quand on veut, au moyen de l'une ou l'autre de ces formules, calculer les nombres S_n^1, S_n^2, ..., S_n^n.

La formule (ζ) où on fait successivement $p = n, n-1, \ldots, 2, 1$, donne pour le calcul de ces nombres les équations

$$\begin{aligned}
&K_{n+1}^n - S_n^1 K_n^n = 0,\\
&K_{n+1}^{n-1} - S_n^1 K_n^{n-1} + S_n^2 K_{n-1}^{n-1} = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&K_{n+1}^2 - S_n^1 K_n^2 + S_n^2 K_{n-1}^2 - \ldots + (-1)^{n-1} S_n^{n-1} K_2^2 = 0,\\
&K_{n+1}^1 - S_n^1 K_n^1 + S_n^2 K_{n-1}^1 - \ldots + (-1)^{n-1} S_n^{n-1} K_2^1 + (-1)^n S_n^n K_1^1 = 0.
\end{aligned}$$

Quant à la formule (11) où on fait $m = 1, 2, \ldots, n-1, n$, et $p = n$, elle donne les équations

$$\begin{aligned}
&K_{n+1}^n - S_n^1 K_n^n = 0,\\
&K_{n+2}^n - S_n^1 K_{n+1}^n + S_n^2 K_n^n = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&K_{2n-1}^n - S_n^1 K_{2n-2}^n + S_n^2 K_{2n-3}^n - \ldots + (-1)^{n-1} S_n^{n-1} K_n^n = 0,\\
&K_{2n}^n - S_n^1 K_{2n-1}^n + S_n^2 K_{2n-2}^n - \ldots + (-1)^{n-1} S_n^{n-1} K_{n+1}^n + (-1)^n S_n^n K_n^n = 0.
\end{aligned}$$

Ces deux systèmes d'équations sont également commodes pour le calcul de S_n^1, S_n^2, ..., S_n^n, mais dans le premier les coefficients sont les nombres du triangle arithmétique de définition, situés dans les n premières colonnes au-dessus

de la $(n+2)^{\text{ième}}$ rangée, dans le second ce sont les nombres de la $n^{\text{ième}}$ colonne situés au-dessus de la $(2n+1)^{\text{ième}}$ rangée.

Il entre ainsi dans le premier système d'équations $\frac{n(n+3)}{2}$ nombres K_m^p et dans le second il n'y en a que $n+1$. Mais il faut remarquer que sur les $\frac{n(n+3)}{2}$ nombres K_m^p qui figurent dans le premier système les nombres $K_n^n, K_{n-1}^{n-1}, \ldots, K_3^3$, d'une part, $K_{n+1}^1, K_n^1, \ldots, K_2^1$ de l'autre sont tous égaux à K_1^1 qui est égal à 1. Il n'y a donc en réalité que $\frac{n(n+3)}{2} - (2n-1)$ ou $\frac{n(n-1)}{2} + 1$ nombres K_m^p distincts parmi les coefficients du premier système, sans compter, bien entendu, les rencontres tout-à-fait fortuites qui peuvent se produire, comme, par exemple, $K_5^2 = K_6^5 = 15$.

Bien que à partir de $n=4$, $\frac{n(n-1)}{2}+1$ soit toujours plus grand que n, le premier système est plus avantageux que le second parce que les nombres K_m^p qui y interviennent sont moins grands et sont, dans la formation du triangle arithmétique, calculés bien avant ceux qui figurent dans le second. Un exemple fera mieux saisir la portée de cette observation. Prenons $n=4$.

Le premier système sera alors

$$\begin{aligned}
10 - s_4^1 &= 0,\\
25 - 6s_4^1 + s_4^2 &= 0,\\
15 - 7s_4^1 + 3s_4^2 - s_4^3 &= 0,\\
1 - s_4^1 + s_4^2 - s_4^3 + s_4^4 &= 0,
\end{aligned}$$

et le second

$$\begin{aligned}
10 - s_4^1 &= 0,\\
65 - 10s_4^1 + s_4^2 &= 0,\\
350 - 65s_4^1 + 10s_4^2 - s_4^3 &= 0,\\
1701 - 350s_4^1 + 65s_4^2 - 10s_4^3 + s_4^4 &= 0.
\end{aligned}$$

Chacun de ces systèmes donne immédiatement par un calcul de proche en proche

$$s_4^1 = 10,\ s_4^2 = 35,\ s_4^3 = 50,\ s_4^4 = 24,$$

mais on voit combien le premier système est plus simple que le second.

7. Je signalerai encore deux remarques relatives aux nombres K_m^p que j'ai été amené à faire, depuis la publication de mon Mémoire, dans d'autres recueils.

La première* consiste en ce que si on répète à propos de la formule $(28)^{\text{bis}}$ de ce Mémoire le raisonnement contenu dans le No. 4 du même travail on arrive à ce théorème:

Si on pose

$$\phi_{m+1}(x) = K_{m+1}^1 + K_{m+1}^2 x + K_{m+1}^3 x^2 + \ldots + K_{m+1}^{m+1} x^m,$$

lorsque l'équation algébrique de degré p, $Z = 0$ *a toutes ses racines réelles, il en est de même de l'équation de degré* $p + m$

$$\phi_{m+1}(x)\, Z + \frac{\phi'_{m+1}(x)}{1} xZ' + \frac{\phi''_{m+1}(x)}{1.2} x^2 Z'' + \ldots + \frac{\phi^{(m)}_{m+1}(x)}{1.2\ldots m} x^m Z^{(m)} = 0.$$

La seconde remarque† est celle-ci:

Si on transforme la formule (54) de mon Mémoire, qui fait connaître les nombres de Bernoulli au moyen des nombres K_m^p,

$$B_m = \frac{K_m^1}{2} - \frac{1!\,K_m^2}{3} + \frac{2!\,K_m^3}{4} - \ldots + (-1)^m \frac{(m-1)!\,K_m^m}{m+1},$$

en faisant usage de la formule (5′) qui donne explicitement K_m^p en fonction de m et p, on obtient

$$B_m = \sum_{\mu=1}^{\mu=m} \left[(-1)^{\mu+1} \mu^{m-1} \sum_{\lambda=\mu}^{\lambda=m} \frac{C_{\lambda-1}^{\mu-1}}{\lambda+1} \right].$$

Cette formule faisant connaître les nombres de Bernoulli en fonction des nombres du triangle arithmétique de Pascal présente un sérieux intérêt théorique, mais, au point-de-vue du calcul effectif des nombres B_m, la formule précédente, lorsqu'on dispose de notre triangle arithmétique des nombres K_m^p, est bien plus expéditive.

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* Comptes-Rendus de l'Académie des Sciences de Paris (1888, 1er Sem., p. 731).

† Bulletin de la Société Mathématique de France (T. XVII, p. 107).

8. Nous croyons utile de donner ici un tableau des nombres K_m^p plus complet que celui qui est contenu dans notre Mémoire.

K_m^p	$p=1$	2	3	4	5	6	7	8	9	10	11	12
$m=1$	1											
2	1	1										
3	1	3	1									
4	1	7	6	1								
5	1	15	25	10	1							
6	1	31	90	65	15	1						
7	1	63	301	350	140	21	1					
8	1	127	966	1701	1050	266	28	1				
9	1	255	3025	7770	6951	2646	462	36	1			
10	1	511	9330	34105	42525	22827	5880	750	45	1		
11	1	1023	28501	145750	246730	179487	63987	11880	1155	55	1	
12	1	2047	86526	611501	1379400	1323652	627396	159027	22275	1705	66	1
...	...	......	....	....	....	....	....	....	......	......	...	...

Sur les lois de forces centrales faisant décrire à leur point d'application une conique quelles que soient les conditions initiales.

PAR P. APPELL.

On sait que les forces centrales, fonctions de la seule position, faisant toujours décrire à leur point d'application une conique ont été déterminées simultanément par MM. Darboux et Halphen* à la suite d'une question posée par M. Bertrand. Sans prétendre rien ajouter d'essentiel à la solution analytique d'Halphen et à l'exposé que M. Tisserand en a fait dans le premier volume de sa *Mécanique céleste*, je me propose d'indiquer rapidement la méthode que j'ai suivie dans mon cours pour abréger autant que possible le calcul d'Halphen. Cette méthode constitue une application de *l'homographie en mécanique* dont j'ai fait la théorie dans ce Journal (1889).

Considérons un mobile de masse 1, sollicité par une force centrale F_1 dépendant seulement des coordonnées (x_1, y_1) de son point d'application par rapport à deux axes rectangulaires O_1x_1, O_1y_1 ayant pour origine le centre O_1 par lequel passe la force.

Les équations du mouvement sont, en appelant le temps t_1,

$$\frac{d^2x_1}{dt_1^2} = F_1\frac{x_1}{r_1}, \quad \frac{d^2y_1}{dt_1^2} = F_1\frac{y_1}{r_1} \tag{1}$$

où

$$r_1 = \sqrt{x_1^2 + y_1^2}.$$

L'intégrale des aires donne

$$x_1\frac{dy_1}{dt_1} - y_1\frac{dx_1}{dt_1} = \alpha.$$

Faisons maintenant la transformation homographique

$$x = \frac{x_1}{y_1}, \quad y = \frac{1}{y_1} \tag{2}$$

* Comptes Rendus, t. 84.

et posons
$$dt = \frac{dt_1}{y_1^2};$$
nous aurons
$$\frac{dx}{dt} = \frac{y\,\dfrac{dx_1}{dt_1} - x_1\,\dfrac{dy_1}{dt_1}}{y_1^2}\cdot\frac{dt_1}{dt} = \alpha, \quad \frac{dy}{dt} = -\frac{1}{y_1^2}\,\frac{dy_1}{dt_1}\,\frac{dt_1}{dt} = -\frac{dy_1}{dt_1},$$
puis
$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = -\frac{d^2y_1}{dt_1^2}\cdot\frac{dt_1}{dt} = -F_1\,\frac{y_1^3}{r_1}. \tag{3}$$
Ces équations montrent que le point (x, y) se meut, dans le temps t, comme un mobile sollicité par une force
$$Y = -F_1\,\frac{y_1^3}{r_1} \tag{4}$$
constamment parallèle à l'axe Oy. Cette force Y est d'ailleurs fonction de x_1 et y_1, et par suite, d'après (2), de x et y. Si le point (x_1, y_1) décrit une conique, le point (x, y) en décrit une autre, transformée homographique de la première, et inversement. Nous sommes donc ramenés à chercher toutes les lois de forces parallèles Y faisant décrire à leur point d'application (x, y) une conique, quelles que soient les conditions initiales. Or ce problème se résout comme il suit.

Les équations du mouvement étant
$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = Y,$$
on a $\dfrac{dx}{dt} = \alpha$ et l'équation différentielle de la trajectoire est
$$\frac{d^2y}{dx^2} = \frac{1}{\alpha^2}\,Y, \tag{5}$$
Y étant une fonction de x et y. Désignons par y', y'', y''', ... les dérivées de y par rapport à x, et rappelons nous que l'équation différentielle des coniques est, d'après Halphen
$$[(y'')^{-\frac{2}{3}}]''' = 0.$$
L'expression (5) de y'' devra vérifier cette équation, quelles que soient les conditions initiales. Soit, en désignant par μ une constante,
$$Y^{-\frac{2}{3}} = \mu^{-\frac{2}{3}}\phi(x, y), \quad Y = \mu\,[\phi(x, y)]^{-\frac{3}{2}} \tag{6}$$

on devra avoir $[\phi(x, y)]''' = 0$. Développant les calculs, on a

$$[\phi(x, y)]' = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} y',$$

$$\phi'' = \frac{\partial^2\phi}{\partial x^2} + 2y' \frac{\partial^2\phi}{\partial x\partial y} + y'^2 \frac{\partial^2\phi}{\partial y^2} + y'' \frac{\partial\phi}{\partial y},$$

$$\phi''' = \frac{\partial^3\phi}{\partial x^3} + 3y' \frac{\partial^3\phi}{\partial x^2\partial y} + 3y'^2 \frac{\partial^3\phi}{\partial x\partial y^2} + y'^3 \frac{\partial^3\phi}{\partial y^3}$$
$$+ 3y'' \left(\frac{\partial^2\phi}{\partial x\partial y} + y' \frac{\partial^2\phi}{\partial y^2}\right) + y''' \frac{\partial\phi}{\partial y}.$$

Comme
$$y'' = \frac{\mu}{\alpha^2} \phi^{-\frac{3}{2}}, \quad y''' = -\frac{3}{2} \frac{\mu}{\alpha^2} \phi^{-\frac{5}{2}} \left(\frac{\partial\phi}{\partial x} + y' \frac{\partial\phi}{\partial y}\right),$$

l'équation $\phi''' = 0$ s'écrit

$$\frac{\partial^3\phi}{\partial x^3} + 3y' \frac{\partial^3\phi}{\partial x^2\partial y} + 3y'^2 \frac{\partial^3\phi}{\partial x\partial y^2} + y'^3 \frac{\partial^3\phi}{\partial y^3} + \frac{3\mu}{2\alpha^2} \phi^{-\frac{5}{2}} \left[2\phi \frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial\phi}{\partial x} \frac{\partial\phi}{\partial y}\right]$$
$$+ \frac{3\mu y'}{2\alpha^2} \phi^{-\frac{5}{2}} \left[2\phi \frac{\partial^2\phi}{\partial y^2} - \left(\frac{\partial\phi}{\partial y}\right)^2\right] = 0.$$

Cette condition devant être remplie quelles que soient les conditions initiales, devra être vérifiée identiquement quels que soient x, y, y' et α, puisque, au commencement du mouvement, ces quatre quantités sont arbitraires. On a donc

$$\frac{\partial^3\phi}{\partial x^3} = 0, \quad \frac{\partial^3\phi}{\partial x^2\partial y} = 0, \quad \frac{\partial^3\phi}{\partial x\partial y^2} = 0, \quad \frac{\partial^3\phi}{\partial y^3} = 0, \tag{7}$$

$$2\phi \frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial\phi}{\partial x} \frac{\partial\phi}{\partial y} = 0, \quad 2\phi \frac{\partial^2\phi}{\partial y^2} - \left(\frac{\partial\phi}{\partial y}\right)^2 = 0. \tag{8}$$

Les condition (7) montrent que ϕ est un polynôme du second degré en x et y

$$\phi(x, y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F.$$

Ce polynôme devant vérifier les conditions (8), deux cas sont à distinguer suivant que C est différent de zéro ou non.

1°. $C \gtrless 0$. Alors $\dfrac{\partial^2\phi}{\partial y^2} = 2C$; la seconde des identités (8) donne

$$\phi = \frac{1}{C} (Bx + Cy + E)^2,$$

expression qui satisfait aussi à la première des identités (8), comme on le vérifie immédiatement.

2°. $C = 0$. Alors, la seconde des identités (8) donne $\dfrac{\partial \phi}{\partial y} = 0$, ϕ ne dépend donc pas de y; l'on a

$$B = C = E = 0,$$
$$\phi = Ax^2 + 2Dx + F;$$

et la première des identités (8) est évidemment satisfaite.

Il y a donc deux lois de forces parallèles répondant à la question: ce sont, d'après (6) les lois exprimées par les formules

$$Y = \frac{\mu C^{\frac{3}{2}}}{(Bx + Cy + E)^3},$$
$$Y = \frac{\mu}{(Ax^2 + 2Dx + F)^{\frac{3}{2}}}.$$

Il y a donc également deux lois de forces centrales répondant à la question. D'après les formules de transformation (2) et (4)

$$x = \frac{x_1}{y_1}, \quad y = \frac{1}{y_1}, \quad F_1 = -\frac{Yr_1}{y_1^3},$$

elles sont données par les formules

$$F_1 = -\frac{\mu r_1 C^{\frac{3}{2}}}{(Bx_1 + Ey_1 + C)^3},$$
$$F_1 = -\frac{\mu r_1}{(Ax_1^2 + 2Dx_1y_1 + Fy_1^2)^{\frac{3}{2}}}.$$

Ce sont bien les deux lois de forces découvertes par MM. Darboux et Halphen.

Remarque. Il est aisé d'étendre les résultats précédents au mouvement d'un point sur une sphère fixe, à l'aide des considérations qui suivent. L'analogie entre les mouvements d'un point sur une sphère et ceux d'un point dans un plan a été signalée depuis longtemps, notamment par M. Paul Serret dans sa thèse: *Sur les propriétés géométriques et mécaniques des lignes à double courbure.* On trouve l'explication de cette analogie dans une transformation semblable à la transformation homographique que nous avons étudiée antérieurement (*American Journal,* t. XII, No. 1). Cette nouvelle transformation fait correspondre, à tout mouvement d'un point dans un plan sous l'action d'une force dépendant uniquement de la position du mobile, le mouvement d'un point sur une sphère sous l'action d'une force dependant uniquement de la position du mobile; et réciproquement.

Etant donnée une sphère (S) de rayon 1 et un plan tangent (P) à cette sphère, nous ferons correspondre, à un point M_1 de la sphère, la projection M de ce point sur le plan (P) faite par le rayon allant du centre au point M_1: c'est la projection bien connue que l'on appelle *centrale* dans la théorie des cartes géographiques; elle fait correspondre à toutes les droites du plan (P) des grands cercles de la sphère (S) et réciproquement. Au point de vue analytique, si l'on prend le point de contact du plan (P) et de la sphère (S) comme pôle d'un système de coordonnées polaires dans le plan et sur la sphère, on aura, en appelant ρ et ω les coordonnées polaires du point M dans le plan, ϕ et θ les coordonnées polaires du point M_1 sur la sphère (ϕ colatitude et θ longitude), les formules de transformation

$$\rho = \operatorname{tang}\phi, \quad \omega = \theta. \tag{a}$$

Si l'on appelle T la demi-force vive d'un point matériel de masse 1 mobile dans le plan (P), on aura

$$T = \frac{1}{2}(\rho'^2 + \rho^2\omega'^2), \quad \rho' = \frac{d\rho}{dt}, \quad \omega' = \frac{d\omega}{dt}$$

et les équations du mouvement seront d'après Lagrange

$$\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\omega}{dt}\right)^2 = R, \quad \frac{d}{dt}\left(\rho^2\frac{d\omega}{dt}\right) = \Omega, \tag{b}$$

R et Ω étant des fonctions de ρ et ω. De même, si l'on appelle T_1 la demi-force vive d'un point sur la sphère, ce point ayant pour masse 1 et se déplaçant pendant le temps t_1, on aura

$$T_1 = \frac{1}{2}(\phi'^2 + \sin^2\phi . \theta'^2), \quad \phi' = \frac{d\phi}{dt_1}, \quad \theta' = \frac{d\theta}{dt_1},$$

et les équations du mouvement de ce point seront

$$\frac{d^2\phi}{dt_1^2} - \sin\phi\cos\phi\left(\frac{d\theta}{dt_1}\right)^2 = \Phi, \quad \frac{d}{dt_1}\left(\sin^2\phi\frac{d\theta}{dt_1}\right) = \Theta, \tag{c}$$

Φ et Θ étant des fonctions de ϕ et θ. Faisons, sur les équations (b) du mouvement plan, la transformation définie par les formules (a) de la projection centrale et établissons entre les temps t et t_1 la relation

$$dt_1 = \cos^2\phi . dt.$$

Nous verrons, par un calcul élémentaire, que les équations (b) prendront la forme (c) où

$$\Phi = \frac{R}{\cos^2\phi}, \quad \Theta = \frac{\Omega}{\cos^2\phi}.$$

Donc à tout mouvement sur le plan correspond un mouvement sur la sphère et réciproquement: la trajectoire de l'un des points est la transformée de la trajectoire de l'autre par projection centrale. Par exemple, si les forces R et Ω sont nulles, le point M décrit une droite dans le plan, les forces Φ et Θ sont nulles aussi et le point M_1 décrit une ligne géodésique de la sphère. Si le point M décrit une conique dans le plan, le point M_1 décrit une conique sphérique et inversement. Pour obtenir toutes les lois de forces dépendant de la position de leur point d'application et faisant décrire à un point mobile sur une sphère une conique sphérique, il suffira donc de trouver dans le plan les lois de forces faisant décrire à leur point d'application une conique (lois trouvées par MM. Darboux et Halphen), puis de leur appliquer la transformation précédente.

On pourrait de même chercher à transformer le mouvement d'un point sur un plan en un mouvement d'un point sur une surface donnée: ce serait là un cas particulier du problème général que nous avons indiqué d'après M. Goursat à la fin du précédent article (*American Journal*, t. XII, p. 114). Pour qu'aux lignes droites du plan correspondent les lignes géodésiques de la surface il faut, d'après un théorème de M. Beltrami, que la surface soit à *courbure constante* (Voyez: Leçons sur la théorie générale des surfaces par M. Darboux, III partie, chapitre III).

On Certain Identities in the Theory of Matrices.

BY HENRY TABER, *Clark University.*

1. In this paper I consider some applications to the general theory of matrices of conceptions familiar in quaternions (the separation into scalar and vector parts, the conjugate, etc.) which I have extended in a previous paper in this Journal, Vol. XII, to matrices of order higher than the second. By this means I find that the identical equation, the identical relations between two matrices (constituting with the identical equation the catena of equations), and Sylvester's formula, may be exhibited as explicit identities, and that the law of latency is an immediate corollary of an explicitly identical proposition. Moreover, by this means I find expressions for the coefficients of the equations of the catena, as simple functions of the sum of the latent roots of the powers and products of powers of the matrices involved. I also prove the extension of the conceptions met with in quaternions without regarding the matrix as an operator linear in and distributive over the units of an algebra, as in establishing these conceptions in the paper above referred to: thus it will be shown that a matrix of the third or of higher order, like a matrix of order two (quaternion), is separable into a scalar and a non-scalar (or vector) part, and that the vector part of a matrix of order ω is further separable into $\omega - 1$ sub-vector parts, which may be termed the first, second, etc., and $(\omega - 1)^{\text{th}}$ vector parts. The separation of a matrix into a scalar and ω vector parts is of much importance in the applications considered in this paper.* In regard to the extension of the conception of the conjugate,† it will be shown that a matrix of order ω has $\omega - 1$ conjugates which

* It will appear later that the identical equation is merely a corollary (immediate) of this separation of a matrix into its ω parts.

† The term *conjugate* is employed in quaternions by Hamilton and Tait with two different significations: 1, to denote a certain function of a quaternion; and 2, to denote a different function, the *converse* (Peirce) or *transverse* (Cayley and Sylvester) of a linear vector operator (matrix of the second or of the third order). (The converse of a given matrix is the matrix obtained by interchanging its rows and

reduce, when $\omega = 2$, to the single conjugate of quaternions; it will appear that the product of a matrix of order ω and its $\omega - 1$ conjugates is a scalar, equal to the content of the matrix; and, following the analogy of quaternions, I define the tensor of the matrix as the ω^{th} root of this product, which is commutative. Further, I shall show that a matrix of order ω may be represented as the product of its tensor into $\omega - 1$ versors, each versor being dependent upon a single parameter contained in all its ω terms, or into a single versor dependent upon $\omega - 1$ parameters, each of which enters into its ω terms. The functions of the parameters which appear in the versors of a matrix are simple extensions of the trigonometrical functions, to which (or to the hyperbolic functions) they reduce when $\omega = 2$.

I shall prove these propositions for matrices of the third order, but the method employed is perfectly general.

2. *A matrix n of the third order, whose latent roots g_1, g_2, g_3 are all distinct, may be represented as follows:*

$$n = \varpi \begin{vmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{vmatrix} \varpi^{-1},$$

*where ϖ is not vacuous.** Let

$$n = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad \varpi = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}, \quad \varpi^{-1} = \begin{vmatrix} B_{11} & B_{21} & B_{31} \\ B_{12} & B_{22} & B_{32} \\ B_{13} & B_{23} & B_{33} \end{vmatrix},$$

columns.) A quaternion may be regarded as a matrix of the second order, and its conjugate (in Hamilton's and Tait's sense) as a matrix or linear vector operator, is not identical with its conjugate as a quaternion; so it is best to restrict the term in quaternions to its first signification. As a term is needed to express the same function of a matrix of order higher than the second, which the conjugate (with the first signification) is of a quaternion, or matrix of the second order, I employ the term conjugate for this purpose.

* From this form of n can obviously be obtained immediately demonstrations of the identical equation, of Sylvester's formula, and of the law of latency. It also gives an expression, more simple than Sylvester's formula, for any function of a matrix whose latent roots are all distinct, provided the function can be expressed in terms of positive integral powers of n, which is not always the case. For then we have

$$Fn = \varpi \begin{vmatrix} Fg_1 & 0 & 0 \\ 0 & Fg_2 & 0 \\ 0 & 0 & Fg_3 \end{vmatrix} \varpi^{-1}.$$

where B_{11}, B_{12}, etc., denote the first minors of ϖ with respect to b_{11}, b_{12}, etc., respectively, divided by the content of ϖ. If we put

$$\left\{\begin{array}{llll}(a_{11}-g_1)\,b_{11}+a_{12}b_{21} & & +\,a_{13}b_{31} & =0,\\ \qquad a_{21}b_{11}+(a_{22}-g_1)\,b_{21} & & +\,a_{23}b_{31} & =0,\\ \qquad a_{31}b_{11}+a_{32}b_{21} & & +\,(a_{33}-g_1)\,b_{31} & =0,\end{array}\right.$$

$$\left\{\begin{array}{llll}(a_{11}-g_2)\,b_{12}+a_{12}b_{22} & & +\,a_{13}b_{32} & =0,\\ \qquad a_{21}b_{12}+(a_{22}-g_2)\,b_{22} & & +\,a_{23}b_{32} & =0,\\ \qquad a_{31}b_{12}+a_{32}b_{22} & & +\,(a_{33}-g_2)\,b_{32} & =0,\end{array}\right.$$

$$\left\{\begin{array}{llll}(a_{11}-g_3)\,b_{13}+a_{12}b_{23} & & +\,a_{13}b_{33} & =0,\\ \qquad a_{21}b_{13}+(a_{22}-g_3)\,b_{23} & & +\,a_{23}b_{33} & =0,\\ \qquad a_{31}b_{13}+a_{32}b_{23} & & +\,(a_{33}-g_3)\,b_{33} & =0,\end{array}\right.$$

the ratio of the constituents of each column of ϖ will be completely determined. From the group of three equations consisting of the first equation of each of these three sets, each transformed by putting the term containing g in the right-hand member, we obtain immediately

$$\begin{aligned}a_{11}&=g_1b_{11}B_{11}+g_2b_{12}B_{12}+g_3b_{13}B_{13},\\ a_{12}&=g_1b_{11}B_{21}+g_2b_{12}B_{22}+g_3b_{13}B_{23},\\ a_{13}&=g_1b_{11}B_{31}+g_2b_{12}B_{32}+g_3b_{13}B_{33},\end{aligned}$$

three equations that, together with the six equations obtained in like manner from the second equations of each of the three sets, and from the third equations of each of the three sets, constitute the conditions necessary and sufficient that

$$n=\varpi\begin{pmatrix}g_1 & 0 & 0\\ 0 & g_2 & 0\\ 0 & 0 & g_3\end{pmatrix}\varpi^{-1}.^{*}$$

If two of the latent roots are equal, as $g_1=g_2$, then n may be represented, in the same way, provided the nullity of $n-g_1$ is two. In general, a matrix of order ω is representable in like manner if its latent roots are all distinct, or if any latent root occurs m times, provided the nullity of the matrix, less that latent root, is m.†

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* Compare this Journal, Vol. XII, p. 359.

† This is readily seen by means of the above scheme of equations determining the constituents of ϖ. It may also be readily proved by regarding a matrix as an operator linear and distributive over the units of an algebra (i. e., as a linear unit, or vector operator). Thus, if of the three latent roots of n, $g_1=g_2$, then n has three linearly independent axes (the only case in which the matrix is representable as above), only if $n-g_1$ annuls two linearly independent vectors; but then $|\,n-g_1\,|$ (the determinant of $n-g_1$) has nullity two. See this Journal, Vol. XII, p. 363.

3. The *first conjugate* of a matrix is obtained from the matrix, when in the form of (2), by a cyclic interchange of its latent roots; the *second conjugate*, by a repetition of this cyclic interchange, etc. Thus the first and second conjugates of the nonion n, denoted respectively by $K_1 n$ and $K_2 n$, are

$$K_1 n = \varpi \begin{pmatrix} g_3 & 0 & 0 \\ 0 & g_1 & 0 \\ 0 & 0 & g_2 \end{pmatrix} \varpi^{-1} \quad K_2 n = \varpi \begin{pmatrix} g_2 & 0 & 0 \\ 0 & g_3 & 0 \\ 0 & 0 & g_1 \end{pmatrix} \varpi^{-1}.$$

Obviously $K_2 n = K_1(K_1 n)$, consequently we may dispense with the subscripts, and write K for K_1, and K^2 for K_2. We have in the case of matrices of the third order, $K.K^2 = K^2.K = K^3 = 1$, whereas in quaternions $K^2 = 1$.

It is very easy to see that the conjugate as defined here is identical, when $\omega = 2$, with the quaternion conjugate. For by (2) any quaternion whose latent roots g_1 and g_2 are distinct, may be represented as

$$q = \varpi \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \varpi^{-1},$$

where ϖ is now a matrix of the second order:

$$\therefore Kq = \varpi \begin{pmatrix} g_2 & 0 \\ 0 & g_1 \end{pmatrix} \varpi^{-1} = \varpi \left(- \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} + 2(g_1 + g_2) \right) \varpi^{-1}$$
$$= -q + 2Sq = Sq - Vq,$$

which is the ordinary definition of the quaternion conjugate.

Evidently, in general for matrices of any order, as in quaternions,

$$K(Fn) = F(Kn).$$

4. If we put

$$\varepsilon_1 = \varpi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \varpi^{-1}, \quad \varepsilon_2 = \varpi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \varpi^{-1}, \quad \varepsilon_3 = \varpi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \varpi^{-1},$$

then

$$n = g_1\varepsilon_1 + g_2\varepsilon_2 + g_3\varepsilon_3,$$
$$Kn = g_3\varepsilon_1 + g_1\varepsilon_2 + g_2\varepsilon_3,$$
$$K^2 n = g_2\varepsilon_1 + g_3\varepsilon_2 + g_1\varepsilon_1,$$

and since the ε's are idempotent and mutually nilfactorial, it follows that any symmetric function of the three conjugate matrices n, Kn, $K^2 n$, (involving no

other matrix but unity) is the same symmetric function of their latent roots into $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$.

From this theorem can immediately be derived the identical equation and Sylvester's formula as explicit identities, and the law of latency may be shown to be involved in an explicit identity:

A. For by the proposition just proved,

$$\begin{aligned} & n^3 - (g_1 + g_2 + g_3)\,n^2 + (g_2g_3 + g_3g_1 + g_1g_2)\,n - g_1g_2g_3 \\ \equiv\ & n^3 - (n + Kn + K^2n)\,n^2 + (Kn.K^2n + K^2n.n + n.Kn)\,n - n.Kn.K^2n \\ \equiv\ & (n - n)(n - Kn)(n - K^2n) \equiv 0 . \end{aligned}$$

B. By Sylvester's formula the expression for any function Fn is

$$\begin{aligned} & \frac{Fg_1.(g_2 - g_3) - Fg_2.(g_1 - g_3) + Fg_3.(g_1 - g_2)}{(g_1 - g_2)(g_1 - g_3)(g_2 - g_3)}\,n^2 \\ & \qquad - \frac{Fg_1.(g_2^2 - g_3^2) - Fg_2.(g_1^2 - g_3^2) + Fg_3.(g_1^2 - g_2^2)}{(g_1 - g_2)(g_1 - g_3)(g_2 - g_3)}\,n \\ & \qquad + \frac{Fg_1.(g_2^2g_3 - g_2g_3^2) - Fg_2.(g_1^2g_3 - g_1g_3^2) + Fg_3.(g_1^2g_2 - g_1g_2^2)}{(g_1 - g_2)(g_1 - g_3)(g_2 - g_3)} . \end{aligned}$$

The coefficients of n^2, n, and 1, are symmetric functions of the g's; hence, for the three latent roots, we may substitute n, Kn, and K^2n, when the expression becomes linear in Fn, $F(Kn)$, and $F(K^2n)$, and, on reducing, the coefficients of $F(Kn)$ and $F(K^2n)$ will appear as zero, and the coefficients of Fn as

$$\frac{[Kn - K^2n]\,n^2 - [(Kn)^2 - (K^2n)^2]\,n + [(Kn)^2.K^2n - Kn.(K^2n)^2]}{(n - Kn)(n - K^2n)(Kn - K^2n)} = 1 .$$

C. Since

$$\begin{aligned} & (Fn)^3 - [Fg_1 + Fg_2 + Fg_3](Fn)^2 + [Fg_2.Fg_3 + Fg_3.Fg_1 + Fg_1.Fg_2](Fn) - Fg_1.Fg_2.Fg_3 \\ \equiv\ & (Fn)^3 - [Fn + F(Kn) + F(K^2n)](Fn)^2 \\ & \quad + [F(Kn).F(K^2n) + F(K^2n).Fn + Fn.F(Kn)](Fn) - Fn.F(Kn).F(K^2n) \\ \equiv\ & (Fn - Fn)(Fn - F(Kn))(Fn - F(K^2n)) \equiv 0 ; \end{aligned}$$

by the principle of this section the latent roots of Fn are found among the same functions of the latent roots of n, namely, Fg_1, Fg_2, Fg_3. It might be, however, that these were not all latent roots of Fn; thus, Fg_1 might occur twice, and Fg_2 once as a latent root of Fn, and we might have

$$(Fn - Fg_1)(Fn - Fg_2) = 0 .$$

In this case the latent roots of $K(Fn)=F(Kn)$ and $K^2(Fn)=F(K^2n)$ would also be Fg_1, occurring twice, and Fg_2: hence

$$Fn+F(Kn)+F(K^2n)=2Fg_1+Fg_2.$$

But the left-hand member is a symmetric function of the latent roots of n equal to $Fg_1+Fg_2+Fg_3$; hence $Fg_1=Fg_3$. In like manner, in any other case, we may show that Fg_1, Fg_2, Fg_3, are all latent roots of Fn.

5. The separation of a quaternion q with distinct latent roots into a scalar and a vector part may be arrived at by putting its latent roots equal, respectively, to $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$, when

$$q=\varpi\begin{pmatrix} a+b\sqrt{-1} & 0 \\ 0 & a-b\sqrt{-1}\end{pmatrix}\varpi^{-1}=a+b.\varpi\begin{pmatrix}\sqrt{-1} & 0 \\ 0 & -\sqrt{-1}\end{pmatrix}\varpi^{-1},$$

i. e.,

$$q=a+bi,$$

where i is a non-scalar square root of -1. It should be noted that $T^2q=a^2+b$ is the content of q.

In the same way, if g_1, g_2, g_3, are the three distinct latent roots of n, and

$$\begin{aligned} g_1 &= a+b+c, \\ g_2 &= a+\lambda b+\lambda^2 c, \\ g_3 &= a+\lambda^2 b+\lambda c, \end{aligned}$$

where λ is an imaginary scalar cube root of unity, then

$$\begin{aligned} n &= a.\varpi\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\varpi^{-1}+b.\varpi\begin{pmatrix}1 & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda^2\end{pmatrix}\varpi^{-1}+c.\varpi\begin{pmatrix}1 & 0 & 0\\ 0 & \lambda^2 & 0\\ 0 & 0 & \lambda\end{pmatrix}\varpi^{-1} \\ &= a+bi+ci^2, \end{aligned}$$

where i is here a non-scalar cube root of unity.* To select the scalar and non-scalar parts of n we may employ the symbols S and V, as in quaternions;

* To complete the analogy between nonions and quaternions, the third power of the nonion units should be scalar imaginary cube roots of unity, just as the quaternion units are fourth roots of unity whose squares are scalars (viz. -1); and in the above separation of the nonion n, i should be a cube root of λ. A similar remark applies to the algebras derived from the vids of a matrix of order higher than the third.

and to distinguish the first and second vector parts, we may employ the symbols V_1 and V_2: thus

$$n = Sn + Vn = Sn + V_1 n + V_2 n.*$$

We then have

$$Kn = Sn + \lambda V_1 n + \lambda^2 V_2 n,$$
$$K^2 n = Sn + \lambda^2 V_1 n + \lambda V_2 n.$$

By definition

$$T^3 n = n.Kn.K^2 n = n.K^2 n.Kn = \text{etc.}\dagger$$
$$= a^3 + b^3 + c^3 - 3abc = S^3 n + T^3 V_1 n + T^3 V_2 n - 3Sn.TV_1 n.TV_2 n.$$

Since, however, $a^3 + b^3 + c^3 - 3abc = g_1 g_2 g_3$, hence $Tn = \sqrt[3]{|n|}$, where $|n|$ denotes the determinant or content of n. Hence it follows that

$$T(nn') = Tn.Tn';$$

and then, if Un denote $n \div Tn$,

$$U(nn') = Un.Un'.$$

6. As an immediate consequence of the separation of a quaternion into a scalar and vector, we have

$$(q - Sq)^2 = V^2 q,$$

which is a form of the identical equation. In matrices of higher orders the identical equation may also be made to appear as an immediate corollary of the separation of a matrix into its ω parts. Thus in the case of a matrix of the third order,

$$(n - a)^3 = (bi + ci^2)^3 = b^3 + c^3 + 3bc\,(bi + ci^2);$$

i. e.,
$$(n - Sn)^3 = T^3 V_1 n + T^3 V_2 n + 3\,TV_1 n.TV_2 n.(n - Sn).$$

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* Neither V_1 nor V_2, as an operator, is distributive over a sum of nonions (as are obviously S and V). I take this occasion to correct an error made in my paper on the theory of matrices in this Journal, Vol. XII, p. 387, where formulae (based on the assumption that V_1 and V_2 are distributive) are given as the nonion analogues of the quaternion formula $V(\alpha\beta + \beta\alpha) = 0$. The true nonion analogue of this formula involves only the symbols S and V. It is given in (8) of this paper.

† In quaternions $K(qq') = Kq'.Kq$. This is not a property of the symbol K for matrices of the third and higher orders. If, however, we define a new conjugate as follows:

$$\mathfrak{K}m = Km.K^2m \ldots . Km^{\omega-1} \div Tm^{\omega-2} = Tm.Um^{-1},$$

which also reduces to the quaternion conjugate when $\omega = 2$, then the property in question of the quaternion conjugate is true also of $\mathfrak{K}$; but neither this conjugate nor K is distributive over a sum of matrices, as is the quaternion conjugate over a sum of quaternions.

For a matrix of the fourth order, $m = a + bi + ci^2 + di^3$, where i is now a non-scalar primitive fourth root of unity,

$$\begin{aligned}(m-a)^4 &= (bi + ci^2 + di^3)^4 \\ &= b^4 - c^4 + d^4 + 4bc^2d - 2b^2d^2 \\ &\qquad + 2(c^2 + 2bd)(m-a)^2 + 4c(b^2 + d^2)(m-a).\end{aligned}$$

In the case of a matrix m of order ω, proceeding in a similar way, the ω^{th} power of the right-hand member (viz. Vm) will be the sum of a scalar ($T^\omega Vm$) and scalar multiples of powers of $Vm = m - Sm$ with exponents from one to $\omega - 2$.

7. On differentiating both members of the identity

$$(q - Sq)^2 = V^2q,$$

we obtain the identical relation between q and any other quaternion $dq = r$, thus:

$$(q - Sq)(r - Sr) + (r - Sr)(q - Sq) = 2SVqVr,$$

i. e.,
$$qr + rq - 2Sq.r - 2Sr.q + 2SqKr = 0.$$

It is readily seen that the scalar and vector parts of the left-hand member are separately zero.

Since the selective symbols V_1 and V_2 are not distributive, it is not possible to proceed in the same way to obtain the identical relation between two nonions from the identical equation in the form given in the last section; the same is true of matrices of higher orders. We may however dispense with the non-distributive selective symbols V_1 and V_2, etc., and employ only S and V, which are distributive for matrices of any order. Thus for the nonion n we have

$$\begin{aligned}VV^3n &= 3bc(bi + ci^2) = \tfrac{3}{2}SV^2n.Vn. \\ \therefore V^3n &= SV^3n + VV^3n = SV^3n + \tfrac{3}{2}SV^2n.Vn,\end{aligned}$$

i. e.,
$$(n - Sn)^3 = SV^3n + \tfrac{3}{2}SV^2n.(n - Sn);$$

and reducing,

$$n^3 - 3Sn.n^2 + 3(S^2n - \tfrac{1}{2}SV^2n)n - (S^3n + SV^3n - \tfrac{3}{2}Sn.SV^2n) = 0.$$

If we replace Vn throughout by $n - Sn$, we get the identical equation freed from the symbol V, viz.

$$n^3 - 3Sn.n^2 + \tfrac{3}{2}(3S^2n - Sn^2)n - (\tfrac{9}{2}S^3n - \tfrac{9}{2}Sn^2.Sn + Sn^3) = 0.$$

Likewise for the matrix m of the fourth order we have

$$V^4m = SV^4m + VV^4m = SV^4m + \tfrac{4}{3}SV^3m.Vm + 2SV^2m.V^2m,$$

i. e.,
$$\begin{aligned} m^4 - 4Sm.m^3 + 6(S^2m - \tfrac{1}{3}SV^2m)\,m^2 \\ - 4(S^3m - SV^2m.Sm + \tfrac{1}{3}SV^3m)\,m \\ + (S^4m - 2S^2m.SV^2m + \tfrac{4}{3}Sm.SV^3m - SV^4m) = 0. \end{aligned}$$

Here also we may substitute throughout in the coefficients $m - Sm$ for Vm.

The same process may be extended to matrices of any order.

Given in either form of this section, it is possible, by successive differentiations of the identical equation of a matrix m of order ω, to obtain the $\omega - 1$ identical relations between m and any other matrix m', which, together with the identical equations in m and m', constitute Sylvester's *catena of equations*. Thus in the case of a matrix of order three, differentiating the identical equation in n, we have, if $dn = n'$,

$$\begin{aligned} (n^2n' + nn'n + n'n^2) - 3Sn.(nn' + n'n) - 3Sn'.n^2 \\ + 3(S^2n - \tfrac{1}{2}SV^2n)\,n' + 3(2SnSn' - SVnVn')\,n \\ - (3S^2nSn' + 3S.V^2nVn' - 3Sn.SVnVn' - \tfrac{3}{2}Sn'.SV^2n) = 0.^* \end{aligned}$$

Freed from the vector symbol, this is

$$\begin{aligned} (n^2n' + nn'n + n'n^2) - 3Sn.(nn' + n'n) - 3Sn'.n^2 \\ + \tfrac{3}{2}(3S^2n - Sn^2)\,n' + 3(3SnSn' - Snn')\,n \\ - (\tfrac{27}{2}S^2n.Sn' - \tfrac{9}{2}Sn^2.Sn' - 9Sn.Snn' + 3Sn^2n') = 0. \end{aligned}$$

Differentiating again with respect to n, regarding n' as a constant and putting the new $dn = n'$, we obtain a new relation between n and n', which, on dividing through by the factor two, is what would be obtained by substituting in the above n for n' and n' for n. Differentiating once again with the same condition as before, we obtain the identical equation in n', on dividing through by the factor three.

8. In the last section it appeared that the identical equation of n could be obtained, by the proper substitution for Vn and the vector of powers of Vn, from the formula for VV^3n, the nonion analogue of the quaternion formula $VV^3q = 0$.†

* Since S and V are distributive, just as in quaternions, $dSn = Sdn$, $dVn = Vdn$. Moreover, as in quaternions, $S(nn') = Sn.Sn' + SVnVn'$, while $V(nn') = Sn.Vn' + Sn'.Vn + V.VnVn'$; and $SVnVn' = SVn'Vn$, so that a cyclic interchange of a scalar product leaves it unaltered.

† See note on page 165 in reference to the nonion analogue of this formula.

On differentiating both members of the nonion analogue of this formula, we obtain

$$V(V^2n.Vn' + Vn.Vn'.Vn + Vn'.V^2n) = \tfrac{8}{3}SV^2n.Vn' + 3SVnVn'.Vn;$$

and differentiating again, regarding n' as a constant,

$$V(Vn.V^2n' + Vn'.Vn.Vn' + V^2n'.Vn) = 3SVnVn'.Vn + \tfrac{8}{3}SV^2n'.Vn.$$

These formulae are the nonion analogues of the quaternion formula

$$V(VqVr + VrVq) = 0,$$

which can be obtained in like manner from $VV^2q = 0$.

Assuming these two formulae, the identical relations between n and n' may be obtained as explicit identities by the proper substitutions in these formulae for Vn, Vn', and the vectors of products of their powers.

If, on the second differentiation of the formula for VV^3n, we put $dn = n''$, still regarding n' as a constant, we shall have

$$V(\Sigma Vn.Vn'.Vn'') = 3(SVn'Vn''.Vn + SVn''Vn.Vn' + SVnVn'.Vn'')$$

(where $\Sigma Vn.Vn'.Vn''$ denotes the sum of the products of Vn, Vn', Vn'', in all possible orders), which is also an analogue of $V(VqVr + VrVq) = 0$. Making the proper substitutions in this formula, we obtain

$$\begin{aligned} &nn'n'' + nn''n' + n'nn'' + n'n''n + n''nn' + n''n'n \\ &- 3Sn.(n'n'' + n''n') - 3Sn'.(n''n + nn'') - 3Sn''.(nn' + n'n) \\ &+ 3(3SnSn' - Snn')n'' + 3(3Sn''Sn - Sn''n)n' + 3(3SnSn' - Snn')n'' \\ &- (27SnSn'Sn'' - 9SnSn'n'' - 9Sn'Sn''n - 9Sn''Snn' + 3Snn'n'' + 3Snn''n') = 0, \end{aligned}$$

which may, of course, also be obtained by differentiating twice the identical equation in n, regarding the first $dn = n'$ as a constant, and putting the second $dn = n''$.

9. The definition of Tn as the cube root of $|n|$ is sufficient to prove that a nonion is separable into the product of a tensor and a versor, since $|nn'| = |n|.|n'|$, and hence the product of two versors is a versor; but, to show the character of the versor part, we may proceed as follows: Since $e^{\theta+\eta}$, $e^{\lambda\theta+\lambda^2\eta}$, $e^{\lambda^2\theta+\lambda\eta}$ may—for proper values of θ and η—have any ratios whatever, we may put

$$n = \varpi\begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix}\varpi^{-1} = \sqrt[3]{g_1g_2g_3}.\varpi\begin{pmatrix} e^{\theta+\eta} & 0 & 0 \\ 0 & e^{\lambda\theta+\lambda^2\eta} & 0 \\ 0 & 0 & e^{\lambda^2\theta+\lambda\eta} \end{pmatrix}\varpi^{-1}.$$

Denoting $\frac{1}{3}\left[e^{\theta+\eta}+e^{\lambda\theta+\lambda^2\eta}+e^{\lambda^2\theta+\lambda\eta}\right]$, $\frac{1}{3}\left[e^{\theta+\eta}+\lambda^2 e^{\lambda\theta+\lambda^2\eta}+\lambda e^{\lambda^2\theta+\lambda\eta}\right]$,
and $\frac{1}{3}\left[e^{\theta+\eta}+\lambda e^{\lambda\theta+\lambda^2\eta}+\lambda^2 e^{\lambda^2\theta+\lambda\eta}\right]$,

by $f_0(\theta, \eta)$, $f_1(\theta, \eta)$, and $f_2(\theta, \eta)$, respectively, then

$$\begin{aligned} n &= Tn.\varpi\begin{pmatrix} f_0(\theta,\eta)+f_1(\theta,\eta)+f_2(\theta,\eta) & 0 & 0 \\ 0 & f_0(\theta,\eta)+\lambda f_1(\theta,\eta)+\lambda^2 f_2(\theta,\eta) & 0 \\ 0 & 0 & f_0(\theta,\eta)+\lambda^2 f_1(\theta,\eta)+\lambda f_2(\theta,\eta)\end{pmatrix}\varpi^{-1} \\ &= Tn.\left(f_0(\theta,\eta)+f_1(\theta,\eta).i+f_2(\theta,\eta).i^2\right), \end{aligned}$$

where $i = V_1 n \div TV_1 n$ is a non-scalar cube root of unity.

If we denote $f_0(\theta, 0) = f_0(0, \theta)$, $f_1(\theta, 0) = f_2(0, \theta)$, $f_2(\theta, 0) = f_1(0, \theta)$ by $f_0\theta$, $f_1\theta$, $f_2\theta$, respectively, it is obvious that

$$n = Tn\,(f_0\theta + f_1\theta.i + f_2\theta.i^2)(f_0\eta + f_2\eta.i + f_1\eta.i^2),^*$$

whence may be derived expressions for the functions $f_0(\theta, \eta)$, etc., in terms of $f_0\theta$, $f_0\eta$, etc., functions of a single parameter.

Since the latent roots of i and i^2 are 1, λ, λ^2, hence, by Sylvester's formula,

$$\begin{aligned} &e^{\theta i} = f_0\theta + f_1\theta.i + f_2\theta.i^2,\quad e^{\eta i^2} = f_0\eta + f_2\eta.i + f_1\eta.i^2, \\ &\therefore n = Tn.e^{\theta i + \eta i^2}. \end{aligned}$$

If, then,

$$\begin{aligned} n' &= Tn'\left(f_0(\theta', \eta') + f_1(\theta', \eta').i + f_2(\theta', \eta').i^2\right) \\ &= Tn'\left(f_0\theta' + f_1\theta'.i + f_2\theta'.i^2\right)\left(f_0\eta' + f_2\eta'.i + f_1\eta'.i^2\right), \\ \therefore nn' &= T(nn')\left(f_0(\theta+\theta', \eta+\eta') + f_1(\theta+\theta', \eta+\eta').i + f_2(\theta+\theta', \eta+\eta').i^2\right) \\ &= T(nn')\left(f_0(\theta+\theta') + f_1(\theta+\theta').i + f_2(\theta+\theta').i^2\right)\left(f_0(\eta+\eta')\right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad \left. + f_2(\eta+\eta').i + f_1(\eta+\eta').i^2\right). \end{aligned}$$

This formula gives the following formulae for the functions of the sum of two arguments:

$$\begin{aligned} f_0(\theta+\theta', \eta+\eta') &= f_0(\theta, \eta).f_0(\theta', \eta') + f_1(\theta, \eta).f_2(\theta', \eta') + f_2(\theta, \eta).f_1(\theta', \eta'), \\ f_1(\theta+\theta', \eta+\eta') &= f_0(\theta, \eta).f_1(\theta', \eta') + f_1(\theta, \eta).f_0(\theta', \eta') + f_2(\theta, \eta).f_2(\theta', \eta'), \\ f_2(\theta+\theta', \eta+\eta') &= f_0(\theta, \eta).f_2(\theta', \eta') + f_2(\theta, \eta).f_0(\theta', \eta') + f_1(\theta, \eta).f_1(\theta', \eta'), \end{aligned}$$

whence may be derived expressions for $f_0(\theta + \theta')$, etc.

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* In the paper referred to in the beginning of this article, in investigating the versor of a nonion, I have considered only $f_0\theta$, $f_0\eta$, etc., functions of a single parameter; but, inadvertently, the above expression is put equal to $Tn\,(f_0(\theta+\eta)+f_1(\theta+\eta).i+f_2(\theta+\eta).i^2)$, which is false.

From the definition of these functions it is evident that

$$f_0(\lambda\theta) = f_0\theta,\ f_1(\lambda\theta) = \lambda f_0\theta,\ f_2(\lambda\theta) = \lambda^2 f_2\theta,$$

$$f_0(\lambda\theta,\ \lambda^2\eta) = f_0(\theta,\ \eta),\ f_1(\lambda\theta,\ \lambda^2\eta) = \lambda f_1(\theta,\ \eta),\ f_2(\lambda\theta,\ \lambda^2\eta) = \lambda^2 f_2(\theta,\ \eta).$$

$$\begin{aligned} \therefore\ Kn &= Tn\left(f_0(\lambda\theta,\ \lambda^2\eta) + f_1(\lambda\theta,\ \lambda^2\eta).i + f_2(\lambda\theta,\ \lambda^2\eta).i^2\right) \\ &= Tn\left(f_0(\lambda\theta) + f_1(\lambda\theta).i + f_2(\lambda\theta).i^2\right)\left(f_0(\lambda^2\eta) + f_2(\lambda^2\eta).i + f_1(\lambda^2\eta).i^2\right), \\ K^2n &= Tn\left(f_0(\lambda^2\theta,\ \lambda\eta) + f_1(\lambda^2\theta,\ \lambda\eta).i + f_2(\lambda^2\theta,\ \lambda\eta).i^2\right) \\ &= Tn\left(f_0(\lambda^2\theta) + f_1(\lambda^2\theta).i + f_2(\lambda^2\theta).i^2\right)\left(f_0(\lambda\eta) + f_2(\lambda\eta).i + f_1(\lambda\eta).i^2\right). \end{aligned}$$

This result may be obtained more simply from the equation $n = Tn.e^{\theta i + \eta . i^2}$ by means of the formula $(K(Fn) = F(Kn)$.

Since $n^{-1} = (Kn.K^2n) \div T^3n$, hence

$$\begin{aligned} n^{-1} &= (Tn)^{-1}\left(f_0(-\theta,\ -\eta) + f_1(-\theta,\ -\eta).i + f_2(-\theta,\ -\eta).i^2\right) \\ &= (Tn)^{-1}\left(f_0(-\theta) + f_1(-\theta).i + f_2(-\theta).i^2\right)\left(f_0(-\eta) + f_2(-\eta).i + f_1(-\eta).i^2\right). \end{aligned}$$

10. In the Phil. Mag., Nov. 1883, Sylvester has given an application of his formula for any function of a matrix to the problem to find an expression for the p^{th} root of a quaternion; his results involve the trigonometric functions. A like process will give the p^{th} root of a nonion in terms of the functions $f_0(\theta,\ \eta)$, etc.; and a similar remark applies to matrices of any order. But Sylvester's result may be obtained in a simplified form and more easily as follows. If g_1 and g_2 are the distinct latent roots of q, then for a proper value of θ and of the quaternion ϖ, we may put

$$q = Tq.\varpi \begin{pmatrix} e^{\theta\sqrt{-1}} & 0 \\ 0 & e^{\theta\sqrt{-1}} \end{pmatrix} \varpi^{-1} = Tq.\varpi \begin{pmatrix} e^{(\theta + 4k_1\pi)\sqrt{-1}} & 0 \\ 0 & e^{-(\theta - 4k_2\pi)\sqrt{-1}} \end{pmatrix} \varpi^{-1}.$$

$$\begin{aligned} \therefore\ q^{\frac{1}{p}} &= T^{\frac{1}{p}}q.\varpi \begin{pmatrix} e^{\left(\frac{\theta}{p} + \frac{4k_1\pi}{p}\right)\sqrt{-1}} & 0 \\ 0 & e^{-\left(\frac{\theta}{p} - \frac{4k_2\pi}{p}\right)\sqrt{-1}} \end{pmatrix} \varpi^{-1} \\ &= T^{\frac{1}{p}}q.e^{\frac{k_1+k_2}{p}2\pi\sqrt{-1}}.\varpi \begin{pmatrix} e^{\left(\frac{\theta}{p} + 2\frac{k_1-k_2}{p}\pi\right)\sqrt{-1}} & 0 \\ 0 & e^{-\left(\frac{\theta}{p} + 2\frac{k_1-k_2}{p}\pi\right)\sqrt{-1}} \end{pmatrix} \varpi^{-1} \\ &= \rho T^{\frac{1}{p}}q.\varpi \begin{pmatrix} \cos\left(\frac{\theta}{p} + \frac{2k}{p}\pi\right) + \sin\left(\frac{\theta}{p} + \frac{2k}{p}\pi\right).\sqrt{-1} & 0 \\ 0 & \cos\left(\frac{\theta}{p} + \frac{2k}{p}\pi\right) - \sin\left(\frac{\theta}{p} + \frac{2k}{p}\pi\right).\sqrt{-1} \end{pmatrix} \varpi^{-1} \\ &= \rho T^{\frac{1}{p}}q\left(\cos\left(\frac{\theta}{p} + \frac{2k}{p}\pi\right) + \sin\left(\frac{\theta}{p} + \frac{2k}{p}\pi\right).UVq\right) \\ &= \rho T^{\frac{1}{p}}q\left(\cos\frac{2k\pi}{p} + \sin\frac{2kp}{p}.UVq\right)\left(\cos\frac{\theta}{p} + \sin\frac{\theta}{p}.UVq\right), \end{aligned}$$

where ρ is a scalar p^{th} root of unity. Whence the p^3 p^{th} roots of q are found among all possible combinations of any p^{th} root of q with the p scalar p^{th} roots of unity and the p quaternion p^{th} roots of unity, whose vector part is parallel to Vq.

From this expression for $q^{\frac{1}{q}}$ we may readily derive

$$q^{\frac{1}{p}} = Aq + B,$$

where

$$A = \rho\, \frac{T^{\frac{1}{p}}q}{Tq}\, \frac{\sin\left(\dfrac{\theta + 2k\pi}{p}\right)}{\sin\theta},$$

$$B = \rho T^{\frac{1}{p}}q \left(\cos\left(\frac{\theta + 2k\pi}{p}\right) - \sin\left(\frac{\theta + 2k\pi}{p}\right)\frac{\cos\theta}{\sin\theta}\right);$$

which are much simpler expressions for A and B than those given by Sylvester in the paper above referred to.

Proceeding in the same way for the nonion n, we have

$$n = Tn.\varpi \begin{pmatrix} e^{\theta + \eta + 6k_1\pi\sqrt{-1}} & 0 & 0 \\ 0 & e^{\lambda\theta + \lambda^2\eta + 6k_2\pi\sqrt{-1}} & 0 \\ 0 & 0 & e^{\lambda^2\theta + \lambda\eta + 6k_3\pi\sqrt{-1}} \end{pmatrix} \varpi^{-1},$$

$$\therefore n^{\frac{1}{p}} = T^{\frac{1}{p}}n.\varpi \begin{pmatrix} e^{\frac{6k_1\pi\sqrt{-1}}{p}} & 0 & 0 \\ 0 & e^{\frac{6k_2\pi\sqrt{-1}}{p}} & 0 \\ 0 & 0 & e^{\frac{6k_3\pi\sqrt{-1}}{p}} \end{pmatrix} \varpi^{-1}.\varpi \begin{pmatrix} e^{\frac{\theta+\eta}{p}} & 0 & 0 \\ 0 & e^{\frac{\lambda\theta + \lambda^2\eta}{p}} & 0 \\ 0 & 0 & e^{\frac{\lambda^2\theta + \lambda\eta}{p}} \end{pmatrix} \varpi^{-1}.$$

If we put

$$\begin{aligned} (\ \ 2k_1 - k_2 - k_3) &= k + k', \\ (-k_1 + 2k_2 - k_3) &= \lambda k + \lambda^2 k', \\ (-k_1 - k_2 + 2k_3) &= \lambda^2 k + \lambda k', \end{aligned}$$

$$\therefore \varpi \begin{pmatrix} e^{\frac{6k_1\pi\sqrt{-1}}{p}} & 0 & 0 \\ 0 & e^{\frac{6k_2\pi\sqrt{-1}}{p}} & 0 \\ 0 & 0 & e^{\frac{6k_3\pi\sqrt{-1}}{p}} \end{pmatrix} \varpi^{-1}$$

$$= e^{\frac{k_1+k_2+k_3}{p}2\pi\sqrt{-1}}\, \varpi \begin{pmatrix} e^{\frac{k+k'}{p}2\pi\sqrt{-1}} & 0 & 0 \\ 0 & e^{\frac{\lambda k + \lambda^2 k'}{p}2\pi\sqrt{-1}} & 0 \\ 0 & 0 & e^{\frac{\lambda^2 k + \lambda k'}{p}2\pi\sqrt{-1}} \end{pmatrix} \varpi^{-1}.$$

Hence, if we put $2k\pi\sqrt{-1} = \varkappa$ and $2k'\pi\sqrt{-1} = \varkappa'$,

$$n^{\frac{1}{p}} = \rho . T^{\frac{1}{p}}n\left({}_0\left(\frac{\varkappa}{p}, \frac{\varkappa'}{p}\right) + f_1\left(\frac{\varkappa}{p}, \frac{\varkappa'}{p}\right).i + f_2\left(\frac{\varkappa}{p}, \frac{\varkappa'}{p}\right).i^2\right.$$
$$\left(f_0\left(\frac{\theta}{p}, \frac{\eta}{p}\right) + f_1\left(\frac{\theta}{p}, \frac{\eta}{p}\right).i + f_2\left(\frac{\theta}{p}, \frac{\eta}{p}\right).i^2\right)$$
$$= \rho . T^{\frac{1}{p}}n\left(f_0\left(\frac{\varkappa+\theta}{p}, \frac{\varkappa'+\eta'}{p}\right) + f_1\left(\frac{\varkappa+\theta}{p}, \frac{\varkappa'+\eta'}{p}\right).i + f_2\left(\frac{\varkappa+\theta}{p}, \frac{\varkappa'+\theta'}{p}\right).i^2\right),$$

where ρ is, as before, a scalar p^{th} root of unity. Consequently the p^3 p^{th} roots of n may all be obtained by combining any p^{th} root of n with the p scalar p^{th} roots of unity and the non-scalar p^{th} roots of unity, whose first and second sub-vector parts are respectively scalar multiples of those of n. From this expression for $n^{\frac{1}{p}}$ it is easy to derive

$$n^{\frac{1}{p}} = An^2 + Bn + C,$$

but the coefficients A, B, C are not very simple expressions.

WORCESTER, MASS., *July* 19, 1890.

Systems of Rays Normal to a Surface.

BY W. C. L. GORTON.

The following article is intended as a supplement to §7 of my paper in this Journal, Vol. X, p. 347. It treats primarily of systems of rays originally passing through a point, which we shall call the focus. The results are not new, and are only given as illustrations of the method employed and the ready applicability of quaternions to such problems. We shall use the laws of reflection and refraction determined by experiment, viz. I. The incident and reflected rays are in the same plane with the normal to the reflecting surface and make equal angles with it; II. The incident and refracted rays are in the same plane with the normal to the refracting surface, and the sines of the angles which they make with it bear a constant ratio to each other. Let $\rho = \sigma + x\tau$ be the equation of a system of rays where $\sigma = f(t, u)$, $\tau = \psi(t, u)$ and $T\tau = 1$. If for any value of x such as $x = \phi(t, u)$ they be normal to a surface, we must have

$$S\tau\partial_t\rho = S\tau\partial_u\rho = 0,$$

where
$$\partial_t\rho = \partial_t\sigma + x\partial_t\tau + \tau\partial_t x$$

and
$$\partial_u\rho = \partial_u\sigma + x\partial_u\tau + \tau\partial_u x,$$

$$\therefore\ S\tau\partial_t\sigma + xS\tau\partial_t\tau + \tau^2\partial_t x = S\tau\partial_t\sigma - \partial_t x = 0,$$
$$S\tau\partial_u\sigma + xS\tau\partial_u\tau + \tau^2\partial_u x = S\tau\partial_u\sigma - \partial_u x = 0,$$

since $S\tau\partial_t\tau = S\tau\partial_u\tau = 0$ and $\tau^2 = -1$, since

$$\frac{d^2x}{dt\,du} = \frac{d^2x}{du\,dt}$$

we must have
$$\frac{\partial}{\partial u} S\tau\partial_t\sigma - \frac{\partial}{\partial t} S\tau\partial_u\sigma = 0,$$

and therefore we obtain as a necessary and sufficient condition

$$S\partial_t\tau\partial_u\sigma - S\partial_u\tau\partial_t\sigma = 0.$$

Let us now consider rays emanating from a point which have been reflected by some surface.

Let $\rho = x\sigma + y\tau$ be the equation of the reflected system where $\rho = x\sigma$ is the equation of the reflecting surface and $T\sigma = T\tau = 1$. Calling ν the normal to the surface $\rho = x\sigma$, since σ and τ are unit vectors, and ν bisects the angle between them, we have

$$\nu \| \tau - \sigma,$$
$$\therefore S(\tau - \sigma)\,\partial_t x\sigma = 0,$$
$$S(\tau - \sigma)\,\partial_u x\sigma = 0,$$

which give us

$$S\tau\partial_t x\sigma + \partial_t x = 0$$

and

$$S\tau\partial_u x\sigma + \partial_u x = 0.$$

Differentiating the first with respect to t and the second with respect to u and subtracting, we have

$$S\partial_t\tau\partial_u x\sigma - S\partial_u\tau\partial_t x\sigma = 0,$$

or the reflected system is normal to some surface. Therefore, in order that a system of rays may be brought to a focus by reflection they must be normal to a surface.

Let $\rho = \sigma + x\tau_1$ be the equation of a normal system of rays; then if the rays be reflected by the surface $\rho = \sigma + x_1\tau_1$ where $x_1 = f(t, u)$, we shall have as the equation of the reflected system,

$$\rho = \sigma + x_1\tau_1 + x\tau,$$

where

$$T\tau_1 = T\tau = 1.$$

Calling ν_1 the normal to the reflecting surface, we have

$$\nu_1 \| \tau_1 - \tau,$$
$$\therefore S(\tau_1 - \tau)\,\partial_t(\sigma + x\tau) = 0,$$
$$S(\tau_1 - \tau)\,\partial_u(\sigma + x\tau) = 0.$$

Treating these as above, we have

$$\begin{aligned} S\partial_t\tau_1\partial_u(\sigma + x\tau) - S\partial_u\tau\partial_t(\sigma + x\tau) &= S\partial_t\tau\partial_u(\sigma + x\tau) - S\partial_u\tau\partial_t(\sigma + x\tau) \\ &= S\partial_t\tau\partial_u\sigma - S\partial_u\tau\partial_t\sigma \\ &= 0, \end{aligned}$$

since the system $\rho = \sigma + x\tau$ was normal to a surface. Therefore, a system of rays which is normal to a surface is still normal to some surface after any number of reflections.

Let $\rho = x\sigma$ be the equation of a system of rays emanating from a point. After reflection by the surface $\rho = x_1\sigma$ their equation will be of the form $\rho = x_1\sigma + y\tau$, and let $\rho = x_1\sigma + y_1\tau$ be the equation of one of the surfaces to which the reflected system is normal. Then since $\tau - \sigma$ is normal to the surface $\rho = x_1\sigma$ and τ to the surface $\rho = x_1\sigma + y_1\tau$, we have

$$S(\sigma - \tau)\,\partial_t x_1\sigma = 0,$$
$$S\tau\partial_t(x_1\sigma + y_1\tau) = 0.$$

Adding these equations we obtain

$$S\sigma\partial_t x_1\sigma + S\tau\partial_t y_1\tau = 0,$$

or

$$\partial_t x + \partial_t y = 0.$$

In the same way we can prove

$$\partial_u x + \partial_u y = 0.$$

Therefore the distance along any ray from the focus to the normal surface is independent of the ray. This can readily be generalized as follows:

Let $$\rho = x_1\tau_1 + x_2\tau_2 + \ldots\ldots + x_n\tau_n + x\tau$$

be the equation of one of the surfaces to which the rays are normal after n reflections. Then, on account of the law of reflection, we have

$$S(\tau_2 - \tau_1)\,\partial_t x_1\tau_1 = 0,$$
$$S(\tau_3 - \tau_2)\,\partial_t(x_1\tau_1 + x_2\tau_2) = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$S(\tau - \tau_n)\,\partial_t(x_1\tau_1 + x_2\tau_2 + \ldots\ldots + x_n\tau_n) = 0.$$

Adding these equations, we obtain

$$S\tau_1\partial_t x_1\tau_1 + \ldots\ldots + S\tau_n\partial_t x_n\tau_n - S\tau\partial_t(x_1\tau_1 + \ldots\ldots + x_n\tau_n) = 0,$$

but

$$S\tau\partial_t(x_1\tau_1 + \ldots\ldots + x_n\tau_n) = -S\tau\partial_t x\tau,$$
$$\therefore\ S\tau_1\partial_t x_1\tau_1 + \ldots\ldots + S\tau_n\partial_t x_n\tau_n + S\tau\partial_t x\tau = 0.$$

Expanding and remembering that $T\tau_1 = \ldots . = T\tau_n = T\tau = 1$, we have

$$\partial_t x_1 + \ldots . + \partial_t x_n + \partial_t x = 0 .$$

In the same way we can prove

$$\partial_u x_1 + \ldots . + \partial_u x_n + \partial_u x = 0 .$$

Therefore, after any number of reflections the distance along any ray from the focus to a normal surface is independent of the ray. These results can readily be extended to the case of refraction with such differences as the difference in the law of refraction introduces.

Let $\rho = x_1\tau_1 + x\tau$ be the equation of a system of rays which emanating from a point have been refracted at the surface $\rho = x_1\tau_1$. Let n_1 and n be the indices of refraction of the two media and $T\tau = T\tau_1 = 1$. By the law of refraction, calling ν the normal to the refracting surface, we have

$$\begin{aligned} &\nu \| n\tau - n_1\tau_1 , \\ \therefore\ &S(n\tau - n_1\tau_1)\,\partial_t x_1\tau_1 = 0 , \\ &S(n\tau - n_1\tau_1)\,\partial_u x_1\tau_1 = 0 ; \end{aligned}$$

expanding

$$\begin{aligned} &nS\tau\partial_t x_1\tau_1 = -n_1\partial_t x_1 , \\ &nS\tau\partial_u x_1\tau_1 = -n_1\partial_u x_1 , \\ \therefore\ &S\partial_t\tau\partial_u x_1\tau_1 - S\partial_u\tau\partial_t x_1\tau_1 = 0 , \end{aligned}$$

and the refracted rays are normal to some surface.

Let $\rho = \sigma + x\tau$ be the equation of any normal system of rays, and let $\rho = \sigma + x\tau + x_1\tau_1$ be the equation of the system after refraction at the surface $\rho = \sigma + x\tau$.

Let m be the index of refraction of the first medium and n that of the second, then

$$\begin{aligned} &S(m\tau - n\tau_1)\,\partial_t(\sigma + x\tau) = 0 , \\ &S(m\tau - n\tau_1)\,\partial_u(\sigma + x\tau) = 0 , \\ &nS\tau_1\partial_t(\sigma + x\tau) = m\partial_t x - mS\tau\partial_t\sigma , \\ &nS\tau_1\partial_u(\sigma + x\tau) = m\partial_u x - mS\tau\partial_u\sigma . \end{aligned}$$

Differentiating the first with respect to u and the second with respect to t and subtracting,

$$m\left(S\partial_u\tau_1\partial_t(\sigma+x\tau)-S\partial_t\tau\partial_u(\sigma+x\tau)\right)=m\left(S\partial_t\tau\partial_u\sigma-S\partial_u\tau\partial_t\sigma\right)$$
$$=0;$$

therefore we have the general theorem that a system of rays normal to a surface are still normal to some surface after any number of reflections and refractions.

Let us consider the equation $\rho=x_1\tau_1+x\tau$ of a surface to which the refracted rays, originally emanating from a point, are normal. Calling the indices of refraction n_1 and n, we have

$$S(n_1\tau_1-n\tau)\,\partial_t x_1\tau_1=0,$$
$$Sn\tau\partial_t(x_1\tau_1+x\tau)=0.$$

Adding, we have
$$n_1S\tau_1\partial_t x_1\tau_1+nS\tau\partial_t x\tau=0,$$

or
$$n_1\partial_t x_1+n\partial_t x=0.$$

In the same way we can prove that

$$n_1\partial_u x_1+n\partial_u x=0.$$

This can be easily generalized as follows:

Let $\rho=x_1\tau_1+x_2\tau_2+\ldots+x_n\tau_n+x\tau$ be the equation of a surface to which the rays are normal after n refractions. Let $n_1, n_2, \ldots n_n, n$ be the indices of refraction, and $T\tau_1=T\tau_2=\ldots=T\tau_n=T\tau=1$. We have, by the laws of refraction,

$$S(n_1\tau_1-n_2\tau_2)\,\partial_t x_1\tau_1=0,$$
$$S(n_2\tau_2-n_3\tau_3)\,\partial_t(x_1\tau_1+x_2\tau_2)=0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$S(n_n\tau_n-n\tau)\,\partial_t(x_1\tau_1+\ldots+x_n\tau_n)=0.$$

Since, then, rays are normal to the above surface, we have

$$S\tau\partial_t(x_1\tau_1+\ldots+x_n\tau_n+x\tau)=0,$$

and therefore
$$nS\tau\partial_t(x_1\tau_1+\ldots+x_n\tau_n)=-nS\tau\partial_t x\tau.$$

Adding the above equations and making use of this relation, we have

$$n_1 S\tau_1 \partial_t x_1 \tau_1 + \ldots + n_n S\tau_n \partial_t x_n \tau_n + nS\tau\partial_t x\tau = 0,$$

or

$$n_1 \partial_t x_1 + \ldots + n_n \partial_t x_n + n\partial_t x = 0.$$

In the same way we can prove

$$n_1 \partial_u x_1 + \ldots + n_n \partial_u x_n + n\partial_u x = 0.$$

Therefore, if we consider the path of any ray from the focus to the normal surface, we have the theorem that sum of the lengths of the path in each medium multiplied by the corresponding index of refraction is independent of the ray.

WOMAN'S COLLEGE, BALTIMORE, *March*, 1890.

On the Epicycloid.

BY F. MORLEY.

The detailed properties of the hypocycloid of class 3 have received much attention since Cremona's paper (Crelle, Vol. 64). See Nouvelles Annales, Series 2, Vol. IX, for papers by Serret, Painvin, Laguerre; Quarterly Journal, IX; Giornale di Mat., XXIII; Messenger, XII (MacMahon); American Journal, X, 3 (Humbert). The peculiar behavior of the general algebraic hypocycloid or epicycloid at infinity being easily determined, few recent mathematicians have stooped to pick up the special properties of these curves. It seems worth while to prove some of these details, using the method of circular coordinates, here very convenient. Some of the results are Wolstenholme's (Proc. London Math. Soc., Vol. 4); most are in line with the very general properties of algebraic curves recently proved by Humbert (referred to later), though the method of proof used in the very simple cases here considered is quite different.

Let, in the epicycloid,

$a \equiv$ radius of fixed or cusp-circle,

$b \equiv$ radius of moving circle,

$c \equiv a + 2b \equiv$ radius of vertex circle,

$\phi \equiv$ inclination of the radius to the point of contact of the circles (a) and (b), the zero line being the radius to a vertex,

$p/q \equiv (c+a)/(c-a)$, p and q being integers prime to each other unless $q = 1$, and p being $> q$.

We have, as in Salmon, Higher Plane Curves, third edition, §305,

$$X = mb \cos\phi + b\cos m\phi,$$
$$Y = mb \sin\phi + b\sin m\phi,$$

where $m \equiv p/q$. Hence writing

$$x \equiv X + iY,$$
$$y \equiv X - iY,$$
$$z \equiv e^{\phi i} \equiv \zeta^q,$$

and noting that $b \equiv (c-a)/2 \equiv qc/(p+q)$, we have

$$\left.\begin{aligned}(p+q)\,x/c &= p\zeta^q + q\zeta^p \\ (p+q)\,y/c &= p\zeta^{-q} + q\zeta^{-p}\end{aligned}\right\}. \qquad (1)$$

Hence the degree of the curve is $2p$.

The tangent is

$$\begin{vmatrix} x & y & 1 \\ p\zeta^q + q\zeta^p & p\zeta^{-q} + q\zeta^{-p} & (p+q)/c \\ pq(\zeta^{q-1} + \zeta^{p-1}) & -pq(\zeta^{-q-1} + \zeta^{-p-1}) & 0 \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} x & y\zeta^{p+q} & 1 \\ p\zeta^q + q\zeta^p & p\zeta^p + q\zeta^q & (p+q)/c \\ 1 & -1 & 0 \end{vmatrix} = 0,$$

or

$$x + y\zeta^{p+q} = c(\zeta^p + \zeta^q). \tag{2}$$

Hence the class of the curve is $p+q$.

For the hypocycloid we have, changing the sign of q since now $c < a$,

$$\left.\begin{aligned} (p-q)\,x/c &= p\zeta^{-q} - q\zeta^p \\ (p-q)\,y/c &= p\zeta^q \;\; - q\zeta^{-p} \end{aligned}\right\}, \tag{3}$$

and

$$x\zeta^q + y\zeta^p = c(1 + \zeta^{p+q}). \tag{4}$$

Hence the degree and class are still $2p$ and $p+q$.

The forms of the curves at infinity are readily got from the above equations, and we see that the only points at infinity are the points I, J, each being counted p times. In the epicycloid the singular tangents at I, J are directed to the origin, where all the foci are collected. The centre being the mean of the foci, the origin is also the centre of the curve. In the hypocycloid the line IJ is the singular tangent at both I and J, and there are no finite foci. See Wolstenholme, Proc. London Math. Soc., IV, p. 327; S. Roberts, ib., pp. 354, 355.

Thus in the hypocycloid of class 4, $p+q=4$, $\therefore p=3$, $q=1$; there are cusps at I, J, and IJ is the common cusp-tangent. One sees that injustice is done to the points IJ by calling this curve (as college text-books are apt to do) the four-cusped hypocycloid.

We may note that for $p+q<5$ or $=6$ there is only one epi- or hypocycloid of given class. I shall distinguish the case $q=1$ (when, since $p/q = (c+a)/(c \sim a) = (a \pm b)/b$, b is a divisor of a and the curve is described in one revolution of moving circle) from the other cases by calling the latter nodal, since there are nodes when $q>1$.

If we draw all the tangents from a point x, y we have from (2)

$$\begin{aligned} \text{Product of } \zeta\text{'s} &= (-)^{p+q} x/y \\ &= (-)^{p+q} e^{2\theta i}, \end{aligned}$$

where θ is the inclination of the radius to x, y. And if θ_r is the inclination of ζ_r,

$$\zeta_r^{p+q} = -e^{2\theta_r i}, \quad r = 1 \text{ to } p+q.$$

Hence

$$(-)^{p+q} e^{2\Sigma\theta_r i} = (-)^{(p+q)^2} e^{2(p+q)\theta i}$$

and

$$\Sigma\theta_r = (p+q)\,\theta.$$

But in the case of the hypocycloid we have in the same way from (4),

$$\Sigma\theta_r = 0.$$

These are cases of Laguerre's theorem (Humbert, American Journal, X, p. 262). They are given by F. R. J. Hervey (Educational Times Reprint, L, p. 110). His method of proof is virtually that used here.

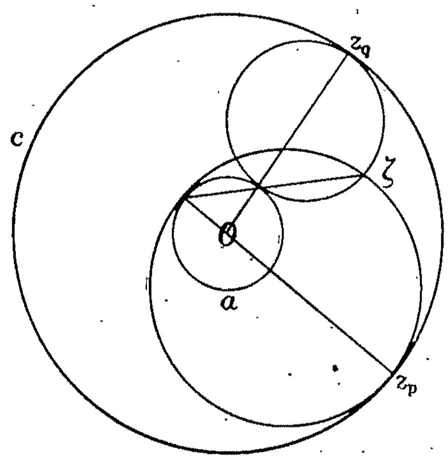
Where (2) meets the vertex circle, $xy = c^2$, so that

$$x^2 + c^2\zeta^{p+q} = cx(\zeta^p + \zeta^q)$$

and

$$\left.\begin{array}{l} x = c\zeta^p \quad \text{or} \quad c\zeta^q \\ y = c\zeta^{-p} \quad \text{or} \quad c\zeta^{-q} \end{array}\right\}. \tag{5}$$

These are the points of contact with the vertex circle of the two generating circles through the point ζ (see fig.).



I shall call the generation by the smaller circle of radius $(c - a)/2$ direct, and that by the larger circle of radius $(c + a)/2$ indirect. A point on the vertex circle is defined by z or $e^{\phi i}$; let the direct point z corresponding to ζ be z_q, the indirect z_p. Then from the equations preceding (1),

$$z_q = \zeta^q, \text{ and similarly } z_p = \zeta^p.$$

Hence there are q direct and p indirect tangents from a point on the vertex circle. For the points of contact of the former, (1) gives

$$(p + q)x/c = pz_q + q\zeta^p,$$

so that they are the vertices of a regular polygon in a circle whose centre is on the radius Oz_q at a distance $pc/(p + q) \equiv (c + a)/2$ from O, and whose radius is $qc/(p + q)$ or $(c - a)/2$; and this is the direct circle which touches the vertex circle at z_q. Hence we have a theorem of Wolstenholme's that *the q direct points*

of contact of tangents from a point z *on the vertex circle form a regular polygon in a direct circle; the same holds for the* p *indirect tangents; and the two circles touch the vertex circle at* z (Wolstenholme, ut supra, p. 322). It is evident that the tangents form two equiangular pencils. This is a familiar fact for the hypocycloid of class 3, in the form that perpendicular tangents meet on the inscribed circle. It is given for the class 4 by R. A. Roberts, Problems on Curves, p. 180.

Start with a point z and draw the direct tangents. They meet the vertex circle again at points z_p such that $z_p = \zeta^p$, $\zeta^q = z$, $z_p = z^{p/q}$.

If we again draw the direct tangents from the points z_p so given, we get new points $z_{pp} = z_p^{p/q} = z^{p^2/q^2}$, etc. Thus the direct tangents from a point on the vertex circle meet the circle again at the vertices of a regular q-gon; the direct tangents from these points meet the circle in a regular q^2-gon and so on. The first q tangents of course form an equiangular system of lines. But any succeeding set has the same property. For $e^{2\theta i} = -\zeta^{p+q}$ from (2). And

$$\zeta^q = z_p = z^{p/q}.$$

Hence

$$e^{2\theta i} = -z^{p(p+q)/q^2}.$$

Hence the angle between two tangents of the second set is $r\pi/q^2$.

This is clearly true for any set of direct or indirect tangents.

If we start with a regular n-gon in the circle, where to avoid coincidence of points n is supposed prime to p and q, we have, if the polygon is given by $z^n = z_0$,

$$z_p = z^{p/q} = z_0^{p/qn},$$

giving a regular qn-gon,

$$z_{pp} = z_p^{p/q} = z_0^{p^2/q^2 n},$$

giving a regular q^2n-gon, and so on.

We have as before for the inclination of any tangent of the first set

$$e^{2\theta i} = -z_0^{(p+q)/nq}.$$

Hence the first set is equiangular, and so in general except when $p+q$ is a multiple of n.

Let, now, $n = p + q$. Then the last equation gives only q values of θ.

Hence, *if from the vertices of a regular* $(p+q)$*-gon in the circle we draw the direct tangents, they form* q *parallel sets.* And from (2) for parallel tangents ζ^{p+q} is given, so that *parallel tangents meet the circle in two regular* $(p+q)$*-gons.*

It is evident geometrically from Wolstenholme's theorem above (see fig.),

or comes at once from equations (1), that each tangent, considered as a chord of the vertex circle, is divided in the ratio q/p at the point of contact.

Hence the points of contact of parallel tangents are the orthogonal projections of a regular polygon.

One immediate inference is the truth for the epicycloid of Chasles' theorem that the mean of the points of contact of parallel tangents is fixed. Others are that the area obtained by joining (in any definite order) the points of contact is constant; and that *the points of contact of parallel tangents lie on an ellipse whose auxiliary circles are the vertex circle and cusp circle, and whose minor axis is parallel to the tangents.*

When tangents are drawn from any point x, y, we have from (2), if $z = \zeta^q$,

$$x + yz^{1+p/q} = c\,(z + z^{p/q}),$$
$$(x - cz)^q = (c - yz)^q z^p,$$

whence, if $p > 1$, $\quad \Sigma\zeta^q = \Sigma z = qc/y.$

Similarly, if $q > 1$, $\quad \Sigma\zeta^p = pc/y.$

Hence, substituting in (1), the mean centre of the points of contact is given by

$$(p+q)^2\,\bar{x}/c = 2pqc/y,$$

and similarly $(p+q)^2\bar{y}/c = 2pqc/x$. Hence *the mean centre of the points of contact of tangents from any point P to a nodal epicycloid is the inverse of P with regard to a fixed circle.*

The radius of the circle is

$$\frac{\sqrt{2pq}}{p+q}\,c, \text{ or } \sqrt{(c^2 - a^2)/2}.$$

A similar calculation for the hypocycloid, starting with (4), gives, if $p > 1$, $q > 1$,

$$\Sigma\zeta^p = px/c,\ \Sigma\zeta^q = qy/c,\ \Sigma\zeta^{-p} = py/c,\ \Sigma\zeta^{-q} = qx/c,$$

and hence from (3), $\bar{x} = \bar{y} = 0$, or *the mean centre of the points of contact from any point to a nodal hypocycloid is fixed.*

This fact is interesting as showing that the enunciation of a theorem of Humbert's (Liouville, 4th series, Vol. III, p. 363) may be generalized. There, among a crowd of important theorems, he states the above fact for any curve which the line infinity meets at points of inflexions but does not touch; but the hypocycloid is not of such a kind.

In the epicycloid the perpendicular from the origin on the tangent is

$$w = 1/2.c.(\zeta^{(p-q)/2} + \zeta^{(q-p)/2}).$$

Now, for all the tangents from a given point we have from (1),

$$\Pi\,(1 - \zeta_r/\zeta) \equiv 1 - c\,(\zeta^{-p} + \zeta^{-q})/y + x\zeta^{-p-q}/y,$$
$$r = 1 \text{ to } p+q.$$

Calculating $\Sigma\zeta_r^{p-q}$ by taking logarithms, we see that when $q > 1$, since $p-q$ is prime to p and q, the sum is zero. So for $\Sigma\zeta_r^{q-p}$. Hence

$$\Sigma w^2 = (p+q)\,c^2/2. \tag{6}$$

Further, if $p-q$ is even, $\Sigma\zeta_r^{(p-q)/2} = 0$, and $\Sigma w = 0$. Hence, *in nodal epicycloids of even class the sum of the distances from the centre on the tangents from any point is zero.*

Since the radius of curvature is proportional to w, such theorems may be also stated for radii of curvature. And since the normal envelops a similar concentric curve, we have, if w' be the perpendicular from the centre on a normal,

$$\Sigma w'^2 = \text{const.},$$

and hence *in a nodal epicycloid the sum of the squares of the distances from the centre to the points of contact of tangents from any point is constant.*

The same statements hold for hypocycloids.

Two tangents cut at a constant angle if the ratio of their parameters ζ is given. Let them be

$$x + y\zeta^{p+q} = c\,(\zeta^p + \zeta^q),$$
$$k^{p+q}x + y\zeta^{p+q} = c\,(k^q\zeta^p + k^p\zeta^q),$$
$$\therefore x\,(1-k^{p+q}) = c\,(1-k^q)\,\zeta^p + c\,(1-k^p)\,\zeta^q,$$
$$y\,(1-k^{p+q}) = ck^q\,(1-k^p)\,\zeta^{-q} + ck^p\,(1-k^q)\,\zeta^{-p},$$

or

$$x = A\zeta_1^p + B\zeta_1^q,$$
$$y = A\zeta_1^{-p} + B\zeta_1^{-q},$$

where

$$\zeta_1 = \zeta/k^{p/2},$$
$$A = ck^{p/2}\,(1-k^q)/(1-k^{p+q}),$$
$$B = ck^{q/2}(1-k^p)(1-k^{p+q}).$$

These are the equations of an epitrochoid in which

$-A =$ distance of generating point from moving centre,

$B =$ sum of radii of fixed and moving circles (Salmon, Curves, §305),

and the ratio in our case of the radii of the fixed and moving circles = that for the epicycloid.

The result is Wolstenholme's (p. 330).

As Wolstenholme indicates (p. 418, see also p. 327), it is not complete when the constant angle is $r\pi/p$ or $r\pi/q$. The vertex circle is then part of the locus. The above equations show this by the vanishing of $1-k^p$ or $1-k^q$. For let the angle be $r\pi/p$. Then

$$k^{p+q} = e^{2r\pi i/p},$$
$$k^{p\,(p+q)} = 1,$$

and one value of k^p is 1, giving the vertex circle. The other values give an epitrochoid.

HAVERFORD COLLEGE, PA.

Reduction of $\frac{dx}{\sqrt{A(1+mx^2)(1+nx^2)}}$ *to* $\frac{Mdy}{\sqrt{(1-y^2)(1-k^2y^2)}}$ *by the Substitution* $x^2 = \frac{a+by^2}{a'+b'y^2}$.

BY H. P. MANNING.

This gives

$$\frac{(ba'-b'a)\,ydy}{\sqrt{A(a+by^2)(a'+b'y^2)[(ma+a')+y^2(mb+b')][(na+a')+y^2(nb+b')]}}.$$

In order that this may assume the desired form one of the factors under the radical must reduce to a constant and another to y^2, and the remaining factors, after division by a constant, must become $1-k^2y^2$ and $1-y^2$. Therefore the four fractions

$$\frac{b}{a},\ \frac{b'}{a'},\ \frac{mb+b'}{ma+a'},\ \frac{nb+b'}{na+a'},$$

are to assume in some order the values

$$0,\ \infty,\ -k^2,\ -1,$$

whose anharmonic ratio is k^2, and since the anharmonic ratio of the four fractions is $\frac{n}{m}$, we have $k^2 = \frac{n}{m}$ or some other of the six values derived from this in the usual way.

(This is evident in another way, for the transformation is the same as transforming $xy(x+my)(x+ny)$ into $xy\,(x-k^2y)(x-y)$ by a homographic transformation which does not alter the anharmonic ratio.)

k^2 must be positive and less than unity; hence

1. If $m<n<0$ or $0<n<m$, we must have $k^2 = \frac{n}{m}$ or $\frac{m-n}{m}$;

2. If $n<m<0$ or $0<m<n$, " " $k^2 = \frac{m}{n}$ or $\frac{n-m}{n}$;

3. If $m<0<n$ or $n<0<m$, " " $k^2 = \frac{m}{m-n}$ or $\frac{n}{n-m}$.

But as we may always take m numerically greater than n when both have the same sign, and positive when they have different signs, all forms will be included in the following four cases:

1. $m < n < 0$ or $0 < n < m$, $k^2 = \frac{n}{m}$;

2. " " " , $k^2 = \frac{m-n}{m}$;

3. $n < 0 < m$, $k^2 = \frac{m}{m-n}$;

4. " , $k^2 = \frac{n}{n-m}$.

To these correspond $[1\,2\,3\,4]$, $[1\,3\,2\,4]$, $[1\,3\,4\,2]$, $[1\,4\,3\,2]$, each in four ways, the position of the figures indicating the order in which the four fractions above assume the values

$$0, \quad \infty, \quad -k^2, \quad -1,$$

and corresponding to these four values we shall get under the radical the four factors

$$1, \; y^2, \; 1-k^2y^2, \; 1-y^2,$$

and the quotient of those two whose position corresponds to the position of 1 and 2 in the brackets, multiplied by a constant factor, say p, will give the substitution for x^2 employed in the case in question. We can then assume the form of the substitution in each case, assign a suitable value to y^2, and determine the value of p quite simply.

For example, in the case $[1\,3\,2\,4]$ we can write

$$x^2 = \frac{a+by^2}{a'+b'y^2} = p\,\frac{1}{1-k^2y^2},$$

and then putting $y^2 = 0$ we get $p = \frac{a}{a'} = -\frac{1}{m}$, since $ma + a' = 0$, therefore the substitution here is $x^2 = -\frac{1}{m(1-k^2y^2)}$.

M is the determinant (ba') or say (12), divided by the square root of A into the numerator which we obtain by reducing our four fractions to a common denominator and adding them. But as one of the four fractions is to become ∞ by the vanishing of its denominator, this factor will be simply the numerator of that fraction multiplied by the other three denominators.

Now if we write these four fractions

$$\frac{b_1}{a_1},\quad \frac{b_2}{a_2},\quad \frac{b_3}{a_3},\quad \frac{b_4}{a_4}$$

in the order in which they are to assume the values

$$0,\quad \infty,\quad -k^2,\quad -1,$$

then

$$M=\sqrt{\frac{(b_1a_2)(b_3a_4)}{Aqa_1b_2a_3a_4}},$$

where

$$(b_1a_2)(b_3a_4)=q\,(12)^2.$$

But

$$(b_1a_2)=-a_1b_2$$

and

$$(b_3a_4)=a_3a_4\begin{vmatrix}-k^2 & -1\\ 1 & 1\end{vmatrix}=a_3a_4\,(1-k^2),$$

$$\therefore M=\sqrt{\frac{-(1-k^2)}{qA}}.$$

Now the six determinants formed by taking the terms of any pair of fractions are connected as follows:

$$(12)=(13)=(14)=-\frac{(23)}{m}=-\frac{(24)}{n}=\frac{(34)}{m-n}.$$

Hence q will always have, except perhaps as to sign, one of the three values

$$-m,\quad -n,\quad m-n.$$

Moreover, the four different arrangements which give the same anharmonic ratio preserve the same pairs, and in each pair the same order, or in both pairs the order is reversed, so that the product will still have the same sign. Therefore the value of q as well as of k^2, and consequently the value of M, is the same for all the four substitutions that come under any one case.

The sign of M, that is, the sign of the radical, can be most easily determined by examining the substitution itself to see what sign it would have on differentiating. I will give the sign in each case in the table.

We have for our four cases:

1. $[1\,2\,3\,4]$, $k^2=\dfrac{n}{m}$, $q=m-n$, $\therefore M=\dfrac{1}{\sqrt{-mA}}$;
2. $[1\,3\,2\,4]$, $k^2=\dfrac{m-n}{m}$, $q=-n$, $M=\dfrac{1}{\sqrt{mA}}$;
3. $[1\,3\,4\,2]$, $k^2=\dfrac{m}{m-n}$, $q=n$, $M=\dfrac{1}{\sqrt{(m-n)\,A}}$;
4. $[1\,4\,3\,2]$, $k^2=\dfrac{n}{n-m}$, $q=m$, $M=\dfrac{1}{\sqrt{-(m-n)\,A}}$.

The second case is applicable both when m, n, and A are all negative and when they are all positive, and it will be more convenient to make five cases instead of four.

I. m, n negative, A positive.
II. m, n, A negative.
III. m, n, A positive.
IV. m, A positive, n negative.
V. m positive, n, A negative.

We can now make a table of all the forms available, and it will be convenient to put $A = \pm 1$ and give the signs of m and n explicitly, so that m and n *in this table* are to be considered positive.

[Those marked with a * are those which Cayley gives. He gives these five cases in the order, taken from Legendre, IV, V, III, I, II.]

I. $(1-mx^2)(1-nx^2)$, $m>n$ $k^2=\frac{n}{m}$ $M=\frac{1}{\sqrt{m}}$ $x^2=$ sign of M	[1234] $\frac{1}{ny^2}$ $-$	[2143] $^*\frac{y^2}{m}$ $+$	[3412] $\frac{1-k^2y^2}{n(1-y^2)}$ $+$	[4321] $\frac{1-y^2}{m(1-k^2y^2)}$ $-$
II. $-(1-mx^2)(1-nx^2)$, $m>n$ $k^2=\frac{m-n}{m}$ $M=\frac{1}{\sqrt{m}}$ $x^2=$ sign of M	[1324] $^*\frac{1}{m(1-k^2y^2)}$ $+$	[2413] $\frac{1-k^2y^2}{n}$ $-$	[3142] $-\frac{y^2}{m(1-y^2)}$ $-$	[4231] $-\frac{1-y^2}{ny^2}$ $+$
III. $(1+mx^2)(1+nx^2)$, $m>n$ $k^2=\frac{m-n}{m}$ $M=\frac{1}{\sqrt{m}}$ $x^2=$ sign of M	[1324] $-\frac{1}{m(1-k^2y^2)}$ $-$	[2413] $-\frac{1-k^2y^2}{n}$ $+$	[3142] $^*\frac{y^2}{m(1-y^2)}$ $+$	[4231] $\frac{1-y^2}{ny^2}$ $-$
IV. $(1+mx^2)(1-nx^2)$ $k^2=\frac{m}{m+n}$ $M=\frac{1}{\sqrt{m+n}}$ $x^2=$ sign of M	[1342] $-\frac{1}{m(1-y^2)}$ $-$	[2431] $^*\frac{1-y^2}{n}$ $-$	[3124] $^*\frac{y^2}{(m+n)(1-k^2y^2)}$ $+$	[4213] $-\frac{(m+n)(1-k^2y^2)}{mny^2}$ $+$
V. $-(1+mx^2)(1-nx^2)$ $k^2=\frac{n}{m+n}$ $M=\frac{1}{\sqrt{m+n}}$ $x^2=$ sign of M	[1432] $^*\frac{1}{n(1-y^2)}$ $+$	[2341] $-\frac{1-y^2}{m}$ $+$	[4123] $-\frac{y^2}{(m+n)(1-k^2y^2)}$ $-$	[3214] $\frac{(m+n)(1-k^2y^2)}{mny^2}$ $-$

A Simple Statement of Proof of Reciprocal-Theorem.

BY J. C. FIELDS.

From the Gaussian Criterion the proof of the Reciprocal-Theorem may be very briefly and directly evolved in the following manner:

Suppose p, q to be relatively prime odd integers and μ the number of least positive remainders (to mod. p) greater than $\frac{1}{2}p$, of the series

$$q,\ 2q,\ 3q,\ \ldots,\ \tfrac{1}{2}(p-1)\,q. \tag{1}$$

Let ν be the like number when p and q are interchanged. We may separate (1) into groups, in each of which the members lie in value between two successive multiples of p, thus:

$$\left.\begin{array}{l} q,\ 2q,\ \ldots \left[\frac{p}{q}\right]q;\ \left(\left[\frac{p}{q}\right]+1\right)q,\ \ldots \left[\frac{2p}{q}\right]q;\ \ldots; \\ \qquad \left(\left[\frac{sp}{q}\right]+1\right)q,\ \ldots \left(\left[\frac{sp}{q}\right]+x\right)q,\ \ldots \left[\frac{(s+1)\,p}{q}\right]q;\ \ldots; \\ \qquad \left(\left[\frac{tp}{q}\right]+1\right)q,\ \ldots;\ \frac{(p-1)\,q}{2}. \end{array}\right\} \tag{2}$$

where in the $(s+1)^{\text{th}}$ group the terms lie in value between sp and $(s+1)\,p$.

In this group any term $\left(\left[\frac{sp}{q}\right]+x\right)q$ gives a least positive remainder $>\frac{1}{2}p$ if $\left(\left[\frac{sp}{q}\right]+x\right)q-sp>\frac{1}{2}p$, i. e. if $x>\frac{(2s+1)\,p}{2q}-\left[\frac{sp}{q}\right]$. The number of terms in the group satisfying this condition is of course the whole number of terms in the group less the greatest value of x for which the condition is not satisfied, i. e.

$$\left(\left[\frac{(s+1)\,p}{q}\right]-\left[\frac{sp}{q}\right]\right)-\left(\left[\frac{(2s+1)\,p}{2q}\right]-\left[\frac{sp}{q}\right]\right)=\left[\frac{(s+1)\,p}{q}\right]-\left[\frac{(2s+1)\,p}{2q}\right].$$

Now in the $(t+1)^{\text{th}}$ and last group, or rather partial group, tp is the greatest multiple of p which is $<\frac{1}{2}(p-1)\,q$, and therefore $t=\left[\frac{(p-1)\,q}{2p}\right]$. Suppose $p>q$, then $t=\frac{1}{2}(q-1)$, $\frac{1}{2}(p-1)\,q-tp=\frac{1}{2}(p-1)\,q-\frac{1}{2}(q-1)\,p<\frac{1}{2}p$.

The last group therefore gives no remainder $>\frac{1}{2}p$, for the last remainder $\frac{1}{2}(p-1)q-tp$, which is the greatest in the group, is $<\frac{1}{2}p$.

Summing up from all the groups the numbers of terms giving remainders $>\frac{1}{2}p$, we get

$$\left.\begin{aligned}\mu &= \sum_s^{t-1}\left(\left[\frac{(s+1)p}{q}\right]-\left[\frac{(2s+1)p}{2q}\right]\right)=\sum_1^{\frac{1}{2}(q-1)}{}_s\left[\frac{sp}{q}\right]-\sum_1^{\frac{1}{2}(q-1)}{}_s\left[\frac{(2s-1)p}{2q}\right]\\ &= \sum_1^{\frac{1}{2}(q-1)}{}_s\left[\frac{sp}{q}\right]-\sum_1^{\frac{1}{2}(q-1)}{}_\sigma\left[\frac{(q-2\sigma)p}{2q}\right]=\sum_1^{\frac{1}{2}(q-1)}{}_s\left[\frac{sp}{q}\right]-\sum_1^{\frac{1}{2}(q-1)}{}_\sigma\left[\frac{p}{2}-\frac{sp}{q}\right]\\ &= -\frac{p-1}{2}\cdot\frac{q-1}{2}+\sum_1^{\frac{1}{2}(q-1)}\left(\left[\frac{sp}{q}\right]-\left[\frac{1}{2}-\frac{sp}{q}\right]\right)\\ &= -\frac{p-1}{2}\cdot\frac{q-1}{2}+\nu+2\sum_1^{\frac{1}{2}(q-1)}{}_s\left[\frac{sp}{q}\right],\end{aligned}\right\} \tag{3}$$

for $\left[\frac{sp}{q}\right]+\left[\frac{1}{2}-\frac{sp}{q}\right]=0$ or -1 according as the least positive remainder obtained on dividing sp by q is $<$ or $>\frac{1}{2}q$, and in our notation the number of such remainders which are $>\frac{1}{2}q$ is ν. We have from (3) therefore

$$\mu+\nu\equiv\frac{p-1}{2}\cdot\frac{q-1}{2}\ (\text{mod. } 2). \tag{4}$$

The above reasoning holds also when p, q are two odd numbers having a greatest common measure r, with the observation that under μ we have reckoned all the terms in (2) which are multiples of p, as giving remainder $p>\frac{1}{2}p$ and not remainder $0<\frac{1}{2}p$. The number of such terms is that of values of s for which $sp=mq$, $m \not> \frac{1}{2}(p-1)$; here $s=\frac{mq}{p}=\frac{mr}{p}\cdot\frac{q}{r}$, and the number of such values of s is evidently $\left[\frac{(p-1)r}{2p}\right]=\frac{r-1}{2}$. If, then, in defining μ we reckon multiples of p as giving a remainder 0, we have simply to substitute $\mu+\frac{1}{2}(r-1)$ for μ in (3), whence

$$\frac{p-1}{2}\cdot\frac{q-1}{2}+\frac{r-1}{2}+\mu-\nu=2\sum_1^{\frac{1}{2}(q-1)}{}_s\left[\frac{sp}{q}\right]$$

and

$$\mu+\nu\equiv\frac{p-1}{2}\cdot\frac{q-1}{2}+\frac{r-1}{2}\ (\text{mod. } 2).$$

Related Expressions for Bernoulli's and Euler's Numbers.

BY J. C. FIELDS.

By the formula*

$$\left(\frac{d}{dx}\right)^n \phi(u) = \sum_1^n{}_r \sum_r^n{}_\rho \frac{(-1)^{\rho-r} u^{\rho-r} \phi^\rho(u)}{r!\,(\rho-r)!} \left(\frac{d}{dx}\right)^n u^r, \tag{1}$$

we may very readily connect Bernoulli's and Euler's numbers.

Put $u = e^{-ix}$, then

$$\sec x + \tan x = \frac{2}{u+i} + i = \sum \frac{E_n x^n}{n!}, \tag{2}$$

where, when n is even, E_n is one of Euler's numbers, and when n is odd, Bernoulli's number $B_{\frac{n+1}{2}} = \frac{(n+1)\,E_n}{2^{n+1}(2^{n+1}-1)}$. Putting $\phi(u) = \frac{1}{u+i}$ in (1), we get

$$\left(\frac{d}{dx}\right)^n \frac{1}{u+i} = \sum_1^n{}_r \sum_r^n{}_\rho \frac{(-1)^r \rho!\, u^{\rho-r}(u+i)^{-(\rho+1)}}{r!\,(\rho-r)!} \left(\frac{d}{dx}\right)^n e^{-rix}, \tag{3}$$

whence

$$\frac{1}{2} E_n = \left(\frac{d}{dx}\right)^n \frac{1}{u+i} = \sum_1^n{}_r \sum_r^n{}_\rho \frac{(-1)^r \rho!\,(-ri)^n (1+i)^{-(\rho+1)}}{r!\,(\rho-r)!}$$

$$= \sum_1^n{}_r \sum_r^n{}_\rho \frac{(-1)^r(-ri)^n}{r!} y^{r+1} \left(\frac{d}{dy}\right)^r y^\rho \Big|_{y=(1+i)^{-1}} = \sum_1^n{}_r \frac{(-1)^r(-ri)^n}{r!} y^{r+1} \left(\frac{d}{dy}\right) \frac{1-y^{n+1}}{1-y}$$

$$= \sum_1^n{}_r \frac{(-1)^r(-ri)^n}{r!} y^{r+1} \left\{ \frac{r!}{(1-y)^{r+1}} - \sum_0^r{}_s r! \binom{n+1}{s} \frac{y^{n-s+1}}{(1-y)^{r-s+1}} \right\}$$

$$= \sum_1^n{}_r (-1)^r(-ri)^n \left(\frac{y}{1-y}\right)^{r+1} \left\{ 1 - y^{n+1} \sum_0^r{}_s \binom{n+1}{s} \left(\frac{1-y}{y}\right)^s \right\}$$

---

*For demonstration of this formula see Bertrand's Calcul Différentiel, p. 141, or *Amer. Journ. of Math.*, Vol. XI, p. 390, formula (7).

$$= y^{n+1}\sum_{1}^{n}(-1)^r(-ri)^n\left(\frac{y}{1-y}\right)^{r+1}\sum_{r+1}^{n+1}\binom{n+1}{s}\left(\frac{1-y}{y}\right)^s$$

$$= y^{n+1}\sum_{2}^{n+1}\sum_{1}^{s-1}\binom{n+1}{s}(-1)^r(-ri)^n\left(\frac{1-y}{y}\right)^{s-r-1}$$

$$= \left(-\frac{1+i}{2}\right)^{n+1}\sum_{2}^{n+1}\sum_{1}^{s-1} i^s\binom{n+1}{s}.\, i^r r^n.$$

Observing that $(1+i)^2 = 2i$, we find

$$\left.\begin{aligned} E_n &= 2\left(\frac{i}{2}\right)^{\frac{n+1}{2}}\sum_{2}^{n+1}\sum_{1}^{s-1} i^{s+r}\binom{n+1}{s}r^n, \quad (n \text{ odd}) \\ E_n &= -(1+i)\left(\frac{i}{2}\right)^{\frac{n}{2}}\sum_{2}^{n+1}\sum_{1}^{s-1} i^{s+r}\binom{n+1}{s}r^n, \quad (n \text{ even}). \end{aligned}\right\} \quad (4)$$

When n is odd we have then

$$B_{\frac{n+1}{2}} = \frac{2(n+1)}{2^{n+1}(2^{n+1}-1)}\left(\frac{i}{2}\right)^{\frac{n+1}{2}}\sum_{2}^{n+1}\sum_{1}^{s-1} i^{s+r}\binom{n+1}{s}r^n.$$

Taking account only of the real terms in the summations on the right side of (4), since E_n being real, the imaginary terms must cancel one another, we may also write (4) under the form

$$\left.\begin{aligned} E_n &= (-1)^{\left[\frac{n}{4}\right]+(-1)^{\frac{n+1}{2}}}2^{-\frac{n+1}{2}} \\ &\quad \sum_{2}^{n+1}\binom{n+1}{s}\left\{\left(1-(-1)^{\frac{n+1}{2}}\right)v_{s-1}-\left(1+(-1)^{\frac{n+1}{2}}\right)v_{s-2}\right\}, \quad (n \text{ odd}) \\ E_n &= (-1)^{\left[\frac{n}{4}\right]}2^{-\frac{n}{2}}\sum_{2}^{n+1}\binom{n+1}{s}\left\{v_{s-1}+(-1)^{\frac{n}{2}}v_{s-2}\right\}, \quad (n \text{ even}) \end{aligned}\right\} \quad (5)$$

where $(-1)^s v_s = s^n - (s-2)^n + (s-4)^n - \ldots.$

Third Memoir on a New Theory of Symmetric Functions.

By Major P. A. MacMahon, R. A.

In this memoir I carry on the development of the Theory of Separations. It is divided into seven sections (8 to 14), and is numbered continuously with the second memoir. It brings the theory up to the point where modes of calculating tables of separations may be advantageously discussed. Sections 8 and 9 lay down the fundamental laws of operation which in sections 10, 11 and 12 are applied to the deduction of some comprehensive theorems of algebraic symmetry. Section 13 is concerned with the multiplication of symmetric functions. Section 14 commences the application of the operators to the functions which appear in a table of separations and establishes the theorem which is preliminary to further researches which may possibly appear in a future number of this journal.

I desire to draw attention to the fundamental theorem in operations of which a statement merely is given in Art. 140 of Section 9. It is a generalization of a theorem of Sylvester, to be found in the Philosophical Magazine, 1877, under the title, "A generalization of Taylor's theorem."

§8.

The Differential Operators.

120. I purpose to keep in view throughout the following investigation the analogy between quantity and operation which was pointed out by Hammond for the first time in Vol. XIII of the Proceedings of the London Mathematical Society.

I present the analogy in an extended form and from two distinct points of view.

Supposing n to be indefinitely great, I write down two relations, viz:

$$1+a_1x+a_2x^2+a_3x^3+\ldots\ldots+a_nx^n$$
$$=(1+\alpha_1x)(1+\alpha_2x)(1+\alpha_3x)\ldots\ldots(1+\alpha_nx),$$
$$1+a_{-1}\frac{1}{x}+a_{-2}\frac{1}{x^2}+a_{-3}\frac{1}{x^3}+\ldots\ldots+a_{-n}\frac{1}{x^n}$$
$$=\left(1+\frac{1}{\alpha_1x}\right)\left(1+\frac{1}{\alpha_2x}\right)\left(1+\frac{1}{\alpha_3x}\right)\ldots\ldots\left(1+\frac{1}{\alpha_nx}\right),$$

which lead to the algebraic theory of the expression of any rational function of the quantities

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots\ldots\ \alpha_n$$

as a rational *integral* function of the number n and of the quantities

$$a_1,\ a_{-1},\ a_2,\ a_{-2},\ a_3,\ a_{-3},\ \ldots\ldots\ a_n,\ a_{-n}.$$

121. By multiplying the two equations together and subsequently multiplying each side by exp n, we obtain

$$e^n\left(1+a_1x+a_2x^2+a_3x^3+\ldots\ldots\right)\left(1+a_{-1}\frac{1}{x}+a_{-2}\frac{1}{x^2}+a_{-3}\frac{1}{x^3}+\ldots\ldots\right)$$
$$=e^n\left(1+\alpha_1x\right)\left(1+\alpha_2x\right)\left(1+\alpha_3x\right)\ldots\ldots\left(1+\frac{1}{\alpha_1x}\right)\left(1+\frac{1}{\alpha_2x}\right)\left(1+\frac{1}{\alpha_3x}\right)\ldots\ldots$$

Observe that the right-hand side is

$$e^n\{1+(1)x+(1^2)x^2+(1^3)x^3+\ldots\ldots\}\left\{1+(\bar{1})\frac{1}{x}+(\bar{1}^2)\frac{1}{x^2}+(\bar{1}^3)\frac{1}{x^3}+\ldots\ldots\right\},$$

that is,

$$e^n\Big[1+(1)x+(\bar{1})\frac{1}{x}+(1^2)x^2+(1)(\bar{1})+(\bar{1}^2)\frac{1}{x^2}+(1^3)x^3+(1^2)(\bar{1})x$$
$$+(1)(\bar{1}^2)\frac{1}{x}+(\bar{1}^3)\frac{1}{x^3}+\ldots\ldots\Big],$$

and this is

$$e^n\left\{1+(1)x+(\bar{1})\frac{1}{x}+(1^2)x^2+(0)+(1\bar{1})+(\bar{1}^2)\frac{1}{x^2}\right.$$
$$\left.+(1^3)x^3+(10)x+(1^2\bar{1})x+(0\bar{1})\frac{1}{x}+(1\bar{1}^2)\frac{1}{x}+(\bar{1}^3)\frac{1}{x^3}+\ldots\ldots\right\}$$
$$=e^n\sum(1^\kappa 0^\lambda \bar{1}^\mu)x^{\kappa-\mu},$$

the summation being for all zero and positive integer values of κ, λ and μ.

But
$$\sum(1^\kappa 0^\lambda \bar{1}^\mu)x^{\kappa-\mu}=\exp\left\{(0)+(1)x+(\bar{1})\frac{1}{x}\right\}$$

symbolically, where

$$\frac{(1)^\kappa(0)^\lambda(\bar{1})^\mu}{\kappa!\,\lambda!\,\mu!} \text{ is the symbolic expression for } (1^\kappa 0^\lambda \bar{1}^\mu).$$

Hence we may write

$$e^n(1 + a_1x + a_2x^2 + \ldots)\left(1 + a_{-1}\frac{1}{x} + a_{-2}\frac{1}{x^2} + \ldots\right)$$
$$= e^n \exp\left\{(0) + (1)x + (\bar{1})\frac{1}{x}\right\} \text{ symbolically.}$$

Returning to the former identity and taking logarithms, we find

$$n + \log(1 + a_1x + a_2x^2 + \ldots) + \log\left(1 + a_{-1}\frac{1}{x} + a_{-2}\frac{1}{x^2} + \ldots\right)$$
$$= (0) + (1)x - \tfrac{1}{2}(2)x^2 + \tfrac{1}{3}(3)x^3 - \ldots$$
$$+ (\bar{1})\frac{1}{x} - \tfrac{1}{2}(\bar{2})\frac{1}{x^2} + \tfrac{1}{3}(\bar{3})\frac{1}{x^3} - \ldots,$$

an identity which indicates that any symmetric function which can be exhibited by means of partitions composed of positive, zero and negative integers, is uniquely expressible by the use of the quantities

$$n, a_1, a_{-1}, a_2, a_{-2}, \ldots;$$

it further gives the law of such expression of the sums of the powers of the quantities

$$\alpha_1, \alpha_2, \alpha_3, \ldots.$$

122. We are now led to the result

$$e^n \exp\left\{(0) + (1)x + (\bar{1})\frac{1}{x}\right\} \text{ symbolic}$$
$$= \exp\left\{\begin{array}{l}(0) + (1)x - \tfrac{1}{2}(2)x^2 + \tfrac{1}{3}(3)x^3 - \ldots \\ \quad + (\bar{1})\frac{1}{x} - \tfrac{1}{2}(\bar{2})\frac{1}{x^2} + \tfrac{1}{3}(\bar{3})\frac{1}{x^3} - \ldots,\end{array}\right\}$$

which gives

$$\exp\left\{(0) + (1)x + (\bar{1})\frac{1}{x}\right\} \text{ symbolic}$$
$$= \exp\left\{\begin{array}{l}\quad (1)x - \tfrac{1}{2}(2)x^2 + \tfrac{1}{3}(3)x^3 - \ldots \\ + (\bar{1})\frac{1}{x} - \tfrac{1}{2}(\bar{2})\frac{1}{x^2} + \tfrac{1}{3}(\bar{3})\frac{1}{x^3} - \ldots\end{array}\right\}$$

a formula of great importance, as will appear presently.

123. Write now

$$\begin{aligned} &1 + A_0 + A_1 x + A_2 x^2 + \ldots\ldots \\ &\qquad + A_{-1}\frac{1}{x} + A_{-2}\frac{1}{x^2} + \ldots\ldots \\ &= \exp\left\{ \begin{aligned} &(0) + (1)x - \tfrac{1}{2}(2)x^2 + \tfrac{1}{3}(3)x^3 - \ldots\ldots \\ &\quad + (\bar{1})\frac{1}{x} - \tfrac{1}{2}(\bar{2})\frac{1}{x^2} + \tfrac{1}{3}(\bar{3})\frac{1}{x^3} - \ldots\ldots \end{aligned} \right\} \end{aligned}$$

so that

$$\begin{aligned} 1 + A_0 &= e^n(1 + a_1 a_{-1} + a_2 a_{-2} + a_3 a_{-3} + \ldots\ldots), \\ A_1 &= e^n(a_1 + a_2 a_{-1} + a_3 a_{-2} + \ldots\ldots), \\ A_{-1} &= e^n(a_{-1} + a_1 a_{-2} + a_2 a_{-3} + \ldots\ldots), \\ A_2 &= e^n(a_2 + a_3 a_{-1} + a_4 a_{-2} + \ldots\ldots), \\ A_{-2} &= e^n(a_{-2} + a_1 a_{-3} + a_2 a_{-4} + \ldots\ldots), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

124. The quantities

$$A_0,\ A_1,\ A_{-1},\ A_2,\ A_{-2},\ \ldots\ldots,$$

now, I believe, introduced into analysis for the first time, are of great and fundamental importance in the theory; subsequently they will be freely adopted as arguments of the symmetric functions, and it will appear that, for the purposes of analysis, they are the proper functions to consider.

125. As I wish to show the complete correspondence which exists between the algebraic theory and the theory of the related partial differential operations, it is necessary to write down some obvious results which, however, are in correspondence with theorems in operations which are by no means obvious.

Thus the last written identity, through expansion of the exponential function and comparison of coefficients, leads to the manifest conclusions:

$$\begin{aligned} 1 + A_0 &= e^{(0)}\Big\{1 + (1)(\bar{1}) + \frac{(1)^2 - (2)}{2!}\cdot\frac{(\bar{1})^2 - (\bar{2})}{2!} \\ &\qquad + \frac{(1)^3 - 3(2)(1) + 2(3)}{3!}\cdot\frac{(\bar{1})^3 - 3(\bar{2})(\bar{1}) + 2(\bar{3})}{3!} + \ldots\ldots\Big\}, \\ A_1 &= e^{(0)}\Big\{(1) + \frac{(1)^2 - (2)}{2!}(\bar{1}) + \frac{(1)^3 - 3(2)(1) + 2(3)}{3!}\cdot\frac{(\bar{1})^2 - (\bar{2})}{2!} + \ldots\ldots\Big\}, \\ A_{-1} &= e^{(0)}\Big\{(\bar{1}) + \frac{(\bar{1})^2 - (\bar{2})}{2!}(1) + \frac{(\bar{1})^3 - 3(\bar{2})(\bar{1}) + 2(\bar{3})}{3!}\cdot\frac{(1)^2 - (2)}{2!} + \ldots\ldots\Big\}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

relations which also are simply reached from the expressions of $A_0, A_1, A_{-1}, \ldots.$ in terms of $n, a_1, a_{-1}, a_2, a_{-2} \ldots.$ by expressing these last quantities in terms of the sums of the powers.

126. Again, from the identity,

$$
\begin{aligned}
1 + A_0 + A_1 x \quad &+ A_2 x^2 \quad + A_3 x^3 \quad + \ldots. \\
+ A_{-1}\frac{1}{x} &+ A_{-2}\frac{1}{x^2} + A_{-3}\frac{1}{x^3} + \ldots. \\
= \exp\Big\{(0) + (1)x \quad &- \tfrac{1}{2}(2)x^2 + (3)\tfrac{1}{3}x^3 - \ldots. \\
+ (\bar{1})\frac{1}{x} &- \tfrac{1}{2}(\bar{2})\frac{1}{x^2} + \tfrac{1}{3}(\bar{3})\frac{1}{x^3} - \ldots.\Big\}
\end{aligned}
$$

we obtain by taking logarithms,

$$
\begin{aligned}
(0) + (1)x \quad &- \tfrac{1}{2}(2)x^2 + \tfrac{1}{3}(3)x^3 - \ldots. \\
+ (\bar{1})\frac{1}{x} &- \tfrac{1}{2}(\bar{2})\frac{1}{x^2} + \tfrac{1}{3}(\bar{3})\frac{1}{x^3} - \ldots. \\
= \log\Big(1 + A_0 + A_1 x \quad &+ A_2 x^2 \quad + A_3 x^3 \quad + \ldots. \\
+ A_{-1}\frac{1}{x} &+ A_{-2}\frac{1}{x^2} + A_{-3}\frac{1}{x^3} + \ldots.\Big)
\end{aligned}
$$

and on expanding and comparing coefficients of like powers of x, we have

$$
\begin{aligned}
(0) = \log(1 + A_0) - \frac{A_1 A_{-1}}{(1+A_0)^2} \qquad & - \frac{A_2 A_{-2}}{(1+A_0)^2} \qquad + \frac{A_1^2 A_{-2}}{(1+A_0)^3} \\
& - \frac{A_3 A_{-3}}{(1+A_0)^2} \qquad + 2\frac{A_2 A_1 A_{-3}}{(1+A_0)^3} \\
- \frac{A_1^3 A_{-3}}{(1+A_0)^4} - \ldots. & + \frac{A_2 A_{-1}^2}{(1+A_0)^3} \qquad - \frac{3}{2}\frac{A_1^2 A_{-1}^2}{(1+A_0)^4} \\
+ 2\frac{A_3 A_{-1} A_{-2}}{(1+A_0)^3} \qquad & - 6\frac{A_2 A_1 A_{-1} A_{-2}}{(1+A_0)^4} + 4\frac{A_1^3 A_{-1} A_{-2}}{(1+A_0)^5} + \ldots. \\
- \frac{A_3 A_{-1}^3}{(1+A_0)^4} \qquad & + 4\frac{A_2 A_1 A_{-1}^3}{(1+A_0)^5} \qquad - \frac{10}{3}\frac{A_1^3 A_{-1}^3}{(1+A_0)^6} - \ldots. \\
= \sum \frac{(-)^{\Sigma\pi - 1}(\Sigma\pi - 1)!}{\ldots \pi_3!\, \pi_2!\, \pi_1!\, \pi_0!\, \pi_{-1}!\, \pi_{-2}!\, \pi_{-3}! \ldots} & \ldots A_3^{\pi_3} A_2^{\pi_2} A_1^{\pi_1} A_0^{\pi_0} A_{-1}^{\pi_{-1}} A_{-2}^{\pi_{-2}} A_{-3}^{\pi_{-3}} \ldots,
\end{aligned}
$$

the summation being in regard to every solution of the indeterminate equation

$$\Sigma t \pi_t = 0$$

in positive, zero and negative integers.

127. In general,

$$\frac{(-)^{m-1}}{m}(m) = \sum \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\ldots\pi_2!\,\pi_1!\,\pi_0!\,\pi_{-1}!\,\pi_{-2}!\ldots} \ldots A_2^{\pi_2}A_1^{\pi_1}A_0^{\pi_0}A_{-1}^{\pi_{-1}}A_{-2}^{\pi_{-2}}\ldots,$$

$$\frac{(-)^{m-1}}{m}(\overline{m}) = \sum \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\ldots\pi_2!\,\pi_1!\,\pi_0!\,\pi_{-1}!\,\pi_{-2}!\ldots} \ldots A_2^{\pi_2}A_1^{\pi_1}A_0^{\pi_0}A_{-1}^{\pi_{-1}}A_{-2}^{\pi_{-2}}\ldots,$$

the summations being in regard to the solutions of the indeterminate equations

$$\Sigma t\pi_t = \quad m,$$
$$\Sigma t\pi_t = -m$$

respectively, in positive, zero and negative integers.

§9.

128. We may express symmetric functions, of the nature here considered, as functions either of the quantities

$$n,\ a_1,\ a_{-1},\ a_2,\ a_{-2},\ \ldots.$$

or of the quantities

$$A_0,\ A_1,\ A_{-1},\ A_2,\ A_{-2},\ \ldots.,$$

where observe that the latter set of arguments does not involve the number n explicitly. Both sets of quantities are defined by the identities

$$e^n(1 + a_1x + a_2x^2 + a_3x^3 + \ldots.)\left(1 + a_{-1}\frac{1}{x} + a_{-2}\frac{1}{x^2} + a_{-3}\frac{1}{x^3} + \ldots.\right)$$
$$= 1 + A_0 + A_1x \quad + A_2x^2 \quad + A_3x^3 \quad + \ldots.$$
$$\qquad + A_{-1}\frac{1}{x} + A_{-2}\frac{1}{x^2} + A_{-3}\frac{1}{x^3} + \ldots.$$
$$= e^n(1+\alpha_1x)(1+\alpha_2x)(1+\alpha_3x)\ldots.\left(1+\frac{1}{\alpha_1x}\right)\left(1+\frac{1}{\alpha_2x}\right)\left(1+\frac{1}{\alpha_3x}\right)\ldots.$$

129. Let us now introduce a quantity μ in addition to the n quantities

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots.\ \alpha_n.$$

We thus obtain the identities

$$e^{n+1}(1+a_1x+a_2x^2+a_3x^3+\ldots)\left(1+a_{-1}\frac{1}{x}+a_{-2}\frac{1}{x^2}+a_{-3}\frac{1}{x^3}+\ldots\right)(1+\mu x)\left(1+\frac{1}{\mu x}\right)$$
$$= \begin{pmatrix} 1 + A_0 + A_1x \quad + A_2x^2 \quad + \ldots. \\ \qquad + A_{-1}\frac{1}{x} + A_{-2}\frac{1}{x^2} + \ldots. \end{pmatrix} e(1+\mu x)\left(1+\frac{1}{\mu x}\right)$$
$$= e^{n+1}(1+\alpha_1x)(1+\alpha_2x)\ldots.(1+\mu x)\left(1+\frac{1}{\alpha_1x}\right)\left(1+\frac{1}{\alpha_2x}\right)\ldots.\left(1+\frac{1}{\mu x}\right),$$

for it is merely necessary to multiply throughout by the factor

$$e(1+\mu x)\left(1+\frac{1}{\mu x}\right).$$

130. The identities may be written

$$e^{n+1}\{1+(a_1+\mu)x+(a_2+a_1\mu)x^2+(a_3+a_2\mu)x^3+\dots\}$$
$$\times\left\{1+\left(a_{-1}+\frac{1}{\mu}\right)\frac{1}{x}+\left(a_{-2}+a_{-1}\frac{1}{\mu}\right)\frac{1}{x^2}+\left(a_{-3}+a_{-2}\frac{1}{\mu}\right)\frac{1}{x^3}+\dots\right\}$$
$$=1+A_0+(2e-1)(1+A_0)+eA_{-1}\mu+eA_1\frac{1}{\mu}$$
$$+A_1+(2e-1)A_1+e(1+A_0)\mu+eA_2\frac{1}{\mu}$$
$$+A_{-1}+(2e-1)A_{-1}+eA_{-2}\mu+e(1+A_0)\frac{1}{\mu}$$
$$+A_2+(2e-1)A_2+eA_1\mu+eA_3\frac{1}{\mu}$$
$$+A_{-2}+(2e-1)A_{-2}+eA_{-3}\mu+eA_{-1}\frac{1}{\mu}$$
$$+\dots\dots\dots\dots$$
$$=e^{n+1}(1+\alpha_1x)(1+\alpha_2x)\dots(1+\mu x)\left(1+\frac{1}{\alpha_1x}\right)\left(1+\frac{1}{\alpha_2x}\right)\dots\left(1+\frac{1}{\mu x}\right),$$

from which it appears that the introduction of the new quantity μ has the effect of changing the quantities

$$\dots a_{-3}\ ,\quad a_{-2}\ ,\quad a_{-1}\ ,\quad n\ ,\quad a_1\ ,\quad a_2\ ,\quad a_3\ ,\dots$$

into

$$a_{-3}+a_{-2}\frac{1}{\mu},\ a_{-2}+a_{-1}\frac{1}{\mu},\ a_{-1}+\frac{1}{\mu},\ n+1,\ a_1+\mu,\ a_2+a_1\mu,\ a_3+a_2\mu,\ \dots$$

respectively, and moreover changes

$$A_0 \text{ into } A_0+(2e-1)(1+A_0)+eA_{-1}\mu+eA_1\frac{1}{\mu},$$
$$A_1 \text{ into } A_1+(2e-1)A_1+e(1+A_0)\mu+eA_2\frac{1}{\mu},$$
$$A_{-1} \text{ into } A_{-1}+(2e-1)A_{-1}+eA_{-2}\mu+e(1+A_0)\frac{1}{\mu},$$
$$A_2 \text{ into } A_2+(2e-1)A_2+eA_1\mu+eA_3\frac{1}{\mu},$$
$$A_{-2} \text{ into } A_{-2}+(2e-1)A_{-2}+eA_{-3}\mu+eA_{-1}\frac{1}{\mu},$$
$$\dots\dots\dots\dots$$

131. Hence if any symmetric function be

$$\phi\{n,\ a_1,\ a_{-1},\ a_2,\ a_{-2},\ a_3,\ a_{-3},\ \ldots\} = \psi\{A_0,\ A_1,\ A_{-1},\ A_2,\ A_{-2},\ \ldots\}$$
$$= \phi = \psi,$$

the altered value will be

$$\phi\left\{n+1,\ a_1+\mu,\ a_{-1}+\frac{1}{\mu},\ a_2+a_1\mu,\ a_{-2}+a_{-1}\frac{1}{\mu},\ \ldots\right\},$$

which by Taylor's theorem is symbolically

$$\exp\left(d_0+d_1\mu+d_{-1}\frac{1}{\mu}\right)\phi,$$

where

$$\begin{aligned} d_0 &= \partial_n, \\ d_1 &= \partial_{a_1} + a_1\partial_{a_2} + a_2\partial_{a_3} + \ldots, \\ d_{-1} &= \partial_{a_{-1}} + a_{-1}\partial_{a_{-2}} + a_{-2}\partial_{a_{-3}} + \ldots, \end{aligned}$$

and the multiplication of operators is symbolic, and will also be

$$\begin{aligned} \psi\Big\{ & A_0 + (2e-1)(1+A_0) + eA_{-1}\mu + eA_1\frac{1}{\mu}, \\ & A_1 + (2e-1)A_1 + e(1+A_0)\mu + eA_2\frac{1}{\mu}, \\ & A_{-1} + (2e-1)A_{-1} + eA_{-2}\mu + e(1+A_0)\frac{1}{\mu}, \ldots\Big\}. \end{aligned}$$

which by Taylor's theorem is symbolically

$$\exp\left\{e\left(g_0+g_1\mu+g_{-1}\frac{1}{\mu}\right)+(e-1)g_0\right\}.\psi,$$

where

$$\begin{aligned} g_0 &= (1+A_0)\partial_{A_0} + A_1\partial_{A_1} + A_2\partial_{A_2} + \ldots \\ &\quad + A_{-1}\partial_{A_{-1}} + A_{-2}\partial_{A_{-2}} + \ldots, \\ g_1 &= (1+A_0)\partial_{A_1} + A_1\partial_{A_2} + A_2\partial_{A_3} + \ldots \\ &\quad + A_{-1}\partial_{A_0} + A_{-2}\partial_{A_{-1}} + \ldots, \\ g_{-1} &= (1+A_0)\partial_{A_{-1}} + A_1\partial_{A_0} + A_2\partial_{A_1} + \ldots \\ &\quad + A_{-1}\partial_{A_{-2}} + A_{-2}\partial_{A_{-3}} + \ldots, \end{aligned}$$

and the multiplication of operators is symbolic.

132. If, moreover, the symmetric function be

$$(pq\ldots o^{\omega}\overline{rs}\ldots) = \phi = \psi,$$

the introduction of the new quantity μ results in the addition to $(pq\ldots o^{\omega}\overline{rs}\ldots)$ of the new terms

$$\begin{aligned} &(q\ldots o^{\omega}\overline{rs}\ldots)\mu^p + (p\ldots o^{\omega}\overline{rs}\ldots)\mu^q + \ldots \\ &+ (pq\ldots o^{\omega-1}\overline{rs})\mu^o + (pq\ldots o^{\omega}\overline{s}\ldots)\frac{1}{\mu^r} + (pq\ldots o^{\omega}\overline{r}\ldots)\frac{1}{\mu^s} + \ldots, \end{aligned}$$

so that we are face to face with the identities

$$\begin{aligned}(pq\ldots\ldots o^{\omega}\overline{rs}\ldots\ldots)+(pq\ldots\ldots o^{\omega-1}\overline{rs})\mu^{o}&\\ +(q\ldots o^{\omega}\overline{rs}\ldots\ldots)\mu^{p}\quad+(p\ldots\ldots o^{\omega}\overline{rs}\ldots\ldots)\mu^{q}&+\ldots\\ +(pq\ldots\ldots o^{\omega}\bar{s}\ldots\ldots)\frac{1}{\mu^{r}}+(pq\ldots\ldots o^{\omega}\bar{r}\ldots\ldots)\frac{1}{\mu^{s}}&+\ldots\\ =\exp\left(d_0+d_1\mu+d_{-1}\frac{1}{\mu}\right).\phi&\\ =\exp\left\{e\left(g_0+g_1\mu+g_{-1}\frac{1}{\mu}\right)+(e-1)g_0\right\}.\psi.&\end{aligned}$$

133. We now write

$$\exp\left(d_0+d_1\mu+d_{-1}\frac{1}{\mu}\right)=1+D_0\mu^0+D_1\mu+D_2\mu^2+\ldots+D_{-1}\frac{1}{\mu}+D_{-2}\frac{1}{\mu^2}+\ldots,$$

$$\exp\left\{e\left(g_0+g_1\mu+g_{-1}\frac{1}{\mu}\right)+(e-1)g_0\right\}=1+G_0\mu^0+G_1\mu+G_2\mu^2+\ldots+G_{-1}\frac{1}{\mu}+G_{-2}\frac{1}{\mu^2}+\ldots,$$

the expansions being of course symbolic, and we find on equating coefficients of like powers of μ,

$$\begin{aligned}D_0\ (pq\ldots\ldots o^{\omega}\overline{rs}\ldots\ldots)&=G_o\ (pq\ldots\ldots o^{\omega}\overline{rs}\ldots\ldots)=(pq\ldots\ldots o^{\omega-1}\overline{rs}\ldots\ldots),\\ D_p\ (pq\ldots\ldots o^{\omega}\overline{rs}\ldots\ldots)&=G_p\ (pq\ldots\ldots o^{\omega}\overline{rs}\ldots\ldots)=(q\ \ldots\ldots o^{\omega}\ \overline{rs}\ldots\ldots),\\ D_q\ (pq\ldots\ldots o^{\omega}\overline{rs}\ldots\ldots)&=G_q\ (pq\ldots\ldots o^{\omega}\overline{rs}\ldots\ldots)=(p\ \ldots\ldots o^{\omega}\ \overline{rs}\ldots\ldots),\\ D_{-r}(pq\ldots\ldots o^{\omega}\overline{rs}\ldots\ldots)&=G_{-r}(pq\ldots\ldots o^{\omega}\overline{rs}\ldots\ldots)=(pq\ldots\ldots o^{\omega}\ \bar{s}\ \ldots\ldots),\\ D_{-s}(pq\ldots\ldots o^{\omega}\overline{rs}\ldots\ldots)&=G_{-s}(pq\ldots\ldots o^{\omega}\overline{rs}\ldots\ldots)=(pq\ldots\ldots o^{\omega}\ \bar{r}\ \ldots\ldots),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

or x, being positive, zero, or negative, D_x or G_x is an operating symbol which, performed upon a monomial symmetric function, has the effect of striking out one part x from its partition.

Also, if P denote a partition which contains no part x,

$$D_xP=G_xP\ =\text{zero}.$$

Moreover, $$D_x(x)=G_x(x)=1.$$

(Compare Hammond, Proceedings of the London Mathematical Society, Vol. XIII, p. 79.)

134. As remarked above, in the two results

$$1 + D_0 + D_1\mu + D_2\mu^2 + \ldots = \exp\left(d_0 + d_1\mu + d_{-1}\frac{1}{\mu}\right)$$
$$+ D_{-1}\frac{1}{\mu} + D_{-2}\frac{1}{\mu^2} + \ldots,$$
$$1 + G_0 + G_1\mu + G_2\mu^2 + \ldots = \exp\left\{e\left(g_0 + g_1\mu + g_{-1}\frac{1}{\mu}\right) + (e-1)\,g_0\right\}$$
$$+ G_{-1}\frac{1}{\mu} + G_{-2}\frac{1}{\mu^2} + \ldots$$

the multiplications arising from the developments of the exponential functions are symbolic.

135. It is convenient to indicate this, as Hammond has done, by placing a horizontal line over the operators which are so multiplied.*

136. We have now the relations which follow, obtained at once by a comparison of coefficients on either side of the two identities

$$1 + D_0 = e^{d_0}\left\{1 + \frac{\overline{d_1 d_{-1}}}{1!\,1!} + \frac{\overline{d_1^2 d_{-1}^2}}{2!\,2!} + \frac{\overline{d_1^3 d_{-1}^3}}{3!\,3!} + \ldots\right\},$$
$$= e^{d_0}\sum_{s=0}^{s=\infty}\frac{\overline{d_1^s d_{-1}^s}}{s!\,s!};$$
$$D_1 = e^{d_0}\left\{d_1 + \frac{\overline{d_1^2 d_{-1}}}{2!\,1!} + \frac{\overline{d_1^3 d_{-1}^2}}{3!\,2!} + \frac{\overline{d_1^4 d_{-1}^3}}{4!\,3!} + \ldots\right\},$$
$$= e^{d_0}\sum_{s=0}^{s=\infty}\frac{\overline{d_1^{s+1} d_{-1}^s}}{(s+1)!\,s!};$$
$$D_{-1} = e^{d_0}\left\{d_{-1} + \frac{\overline{d_1 d_{-1}^2}}{1!\,2!} + \frac{\overline{d_1^2 d_{-1}^3}}{2!\,3!} + \frac{\overline{d_1^3 d_{-1}^4}}{3!\,4!} + \ldots\right\},$$
$$= e^{d_0}\sum_{s=0}^{s=\infty}\frac{\overline{d_1^s d_{-1}^{s+1}}}{s!\,(s+1)!};$$

* It will be convenient also in future to write $\overline{\exp}\, u$ when the multiplication of operators that occur in u is symbolic, and $\exp u$ in other cases.

and in general

$$D_\kappa = e^{d_0}\sum_{s=0}^{s=\infty}\frac{\overline{d_1^{s+\kappa}d_{-1}^s}}{(s+\kappa)!\ s!};$$

$$D_{-\kappa} = e^{d_0}\sum_{s=0}^{s=\infty}\frac{\overline{d_1^s d_{-1}^{s+\kappa}}}{s!\ (s+\kappa)!};$$

whatever be the value of κ.

137. Also

$$1+G_0 = e^{(2e-1)g_0}\sum_{s=0}^{s=\infty}e^{2s}\frac{\overline{g_1^s g_{-1}^s}}{s!\ s!},$$

$$G_\kappa = e^{(2e-1)g_0}\sum_{s=0}^{s=\infty}e^{2s+\kappa}\frac{\overline{g_1^{s+\kappa}g_{-1}^s}}{(s+\kappa)!\ s!},$$

$$G_{-\kappa} = e^{(2e-1)g_0}\sum_{s=0}^{s=\infty}e^{2s+\kappa}\frac{\overline{g_1^s g_{-1}^{s+\kappa}}}{s!\ (s+\kappa)!}.$$

138. In these relations the factors $\overline{\exp}\, d_0$, $\overline{\exp}\,(2e-1)\,g_0$ are to be multiplied symbolically into all that follows; this is of no importance in the case of the factor $\overline{\exp}\, d_0$, because neither of the operations d_1, d_{-1} involves the quantity n as a symbol of quantity; we may in this case, if we choose, perform the operations

$$\left\{1+\frac{\overline{d_1 d_{-1}}}{1!\ 1!}+\ldots\right\}$$

and

$$\exp d_0 = 1+\frac{\partial_n}{1!}+\frac{\partial_n^2}{2!}+\ldots$$

successively, for their successive operation is equivalent to their symbolic multiplication. Also the symbols which occur in d_1 and d_{-1} are altogether independent, so that the operation

$$\overline{d_1^{s+\kappa}d_{-1}^s}$$

may be regarded as the successive performance of the operations

$$\overline{d_1^{s+\kappa}}$$

and

$$\overline{d_{-1}^s},$$

and may likewise be written

$$\overline{d_1^{s+\kappa}}\ \overline{d_{-1}^s};$$

hence we may also write

$$D_\kappa = \sum_{t=0}^{t=\infty} \sum_{s=0}^{s=\infty} \frac{\overline{d_1^{s+\kappa} d_0^t d_{-1}^s}}{(s+\kappa)!\, t!\, s!},$$

$$D_{-\kappa} = \sum_{t=0}^{t=\infty} \sum_{s=0}^{s=\infty} \frac{\overline{d_1^s d_0^t d_{-1}^{s+\kappa}}}{s!\, t!\, (s+\kappa)!}.$$

With regard to the operations g_0, g_1, g_{-1} the case is different, since the same quantities occur as symbols of quantity in all three. Hence it is absolutely necessary to actually perform all the symbolic multiplications.

139. It is now necessary to find the expressions for

$$\overline{\exp}\left(d_0 + d_1\mu + d_{-1}\frac{1}{\mu}\right)$$

and

$$\overline{\exp}\left\{e\left(g_0 + g_1\mu + g_{-1}\frac{1}{\mu}\right) + (e-1)\, g_0\right\}$$

in series of products of linear operators in which the multiplication is not symbolic.

In the first place write

$$d_\lambda = \partial_{a_\lambda} + a_1\partial_{a_{\lambda+1}} + a_2\partial_{a_{\lambda+2}} + \ldots .$$

$$d_{-\lambda} = \partial_{a_{-\lambda}} + a_{-1}\partial_{a_{-\lambda-1}} + a_{-2}\partial_{a_{-\lambda-2}} + \ldots .$$

Now

$$\overline{\exp}\, d_1\mu = 1 + d_1\mu + \frac{\overline{d_1^2}}{2!}\mu^2 + \frac{\overline{d_1^3}}{3!}\mu^3 + \ldots ,$$

and the correspondence between the algebra of the operators d_λ and that of symmetric functions was pointed out and established by HAMMOND (loc. cit. Proc. Lond. Math. Soc.).

Hence since, as is well known,

$$1 + (1)\mu + (1^2)\mu^2 + (1^3)\mu^3 + \ldots = \exp\{(1)\mu - \tfrac{1}{2}(2)\mu^2 + \tfrac{1}{3}(3)\mu^3 - \ldots\}$$

we derived at once

$$1 + d_1\mu + \frac{\overline{d_1^2}}{2!}\mu^2 + \frac{\overline{d_1^3}}{3!}\mu^3 + \ldots = \exp(d_1\mu - \tfrac{1}{2}d_2\mu^2 + \tfrac{1}{3}d_3\mu^3 - \ldots);$$

that is,

$$\overline{\exp}\, d_1\mu = \exp(d_1\mu - \tfrac{1}{2}d_2\mu^2 + \tfrac{1}{3}d_3\mu^3 - \ldots),$$

where the multiplications on the dexter are not symbolic.

Similarly

$$\overline{\exp}\, d_{-1}\frac{1}{\mu} = \exp\left(d_{-1}\frac{1}{\mu} - \tfrac{1}{2} d_{-2}\frac{1}{\mu^2} + \tfrac{1}{3} d_{-3}\frac{1}{\mu^3} - \ldots\right).$$

Hence, remarking what has gone before, the absolute identity

$$\overline{\exp}\left(d_0 + d_1\mu + d_{-1}\frac{1}{\mu}\right) = \exp\left\{\begin{array}{l} d_0 + d_1\mu - \tfrac{1}{2} d_2\mu^2 + \tfrac{1}{3} d_3\mu^3 - \ldots \\ \qquad + d_{-1}\frac{1}{\mu} - \tfrac{1}{2} d_{-2}\frac{1}{\mu^2} + \tfrac{1}{3} d_{-3}\frac{1}{\mu^3} - \ldots \end{array}\right\}.$$

140. In the second place, writing in general

$$\begin{array}{l} g_\lambda \doteq (1 + A_0)\,\partial_{A_\lambda} + A_1\partial_{A_{\lambda+1}} + A_2\partial_{A_{\lambda+2}} + \ldots \\ \qquad\qquad + A_{-1}\partial_{A_{\lambda-1}} + A_{-2}\partial_{A_{\lambda-2}} + \ldots, \end{array}$$

where λ is any positive, zero, or negative integer, it will be convenient to at once put on record a theorem of great generality and importance that I have been led to by the present investigation.

Let

$$\exp\left\{\begin{array}{l} f_0 + f_1 y + f_2 y^2 + \ldots \\ + f_{-1}\frac{1}{y} + f_{-2}\frac{1}{y^2} + \ldots \end{array}\right\} = \begin{array}{l} 1 + F_0 + F_1 y + F_2 y^2 + \ldots \\ \qquad + F_{-1}\frac{1}{y} + F_{-2}\frac{1}{y^2} + \ldots, \end{array}$$

where y is arbitrary, be an absolute identity. Then the theorem asserts the absolute identity

$$\exp\left(\begin{array}{l} f_0 g_0 + f_1 g_1 + f_2 g_2 + \ldots \\ + f_{-1} g_{-1} + f_{-2} g_{-2} + \ldots \end{array}\right) = \overline{\exp}\left(\begin{array}{l} F_0 g_0 + F_1 g_1 + F_2 g_2 + \ldots \\ \qquad + F_{-1} g_{-1} + F_{-2} g_{-2} + \ldots \end{array}\right).$$

Of this theorem I have two independent proofs which will be communicated elsewhere; it enables us from any linear function P of the operators to determine another linear function Q such that

$$\exp P \doteq \overline{\exp}\, Q,$$

or conversely.

141. In the case before us we are given

$$\overline{\exp}\left\{(2e - 1)\, g_0 + e\mu g_1 + \frac{e}{\mu} g_{-1}\right\},$$

so that $F_0 = 2e - 1$, $F_1 = e\mu$, $F_{-1} = \dfrac{e}{\mu}$ and the remainder of the F's vanish.

Hence taking logarithms of the ruling identity

$$\begin{aligned} & f_0 + f_1 y + f_2 y^2 + \ldots = \log\left(2e + e\mu y + \frac{e}{\mu y}\right) \\ & \quad + f_{-1}\frac{1}{y} + f_{-2}\frac{1}{y^2} + \ldots \\ & = 1 + \log(1+\mu y) + \log\left(1 + \frac{1}{\mu y}\right) \\ & = 1 + \mu y - \tfrac{1}{2}\mu^2 y^2 + \tfrac{1}{3}\mu^3 y^3 - \ldots \\ & \quad + \frac{1}{\mu y} - \tfrac{1}{2}\frac{1}{\mu^2 y^2} + \tfrac{1}{3}\frac{1}{\mu^3 y^3} - \ldots, \end{aligned}$$

and thus $f_0, f_1, f_{-1}, f_2, f_{-2}, \ldots$ are determined, and we reach the relation

$$\begin{aligned} & \exp\left(\begin{aligned} & g_0 + g_1\mu - \tfrac{1}{2}g_2\mu^2 + \tfrac{1}{3}g_3\mu^3 - \ldots \\ & + g_{-1}\frac{1}{\mu} - \tfrac{1}{2}g_{-2}\frac{1}{\mu^2} + \tfrac{1}{3}g_{-3}\frac{1}{\mu^3} - \ldots \end{aligned}\right) \\ & \qquad = \overline{\exp}\left\{(2e-1)g_0 + eg_1\mu + eg_{-1}\frac{1}{\mu}\right\}. \end{aligned}$$

142. For present purposes it is useful to mention another theorem in regard to these operations.

Let P denote any linear function of the operators and consider the combination

$$\exp g_0 \,.\, \overline{\exp}\, P,$$

where $\overline{\exp}\, P$ and $\exp g_0$ denote two successive operations.

It is easy to prove that

$$\exp g_0 \,.\, \overline{\exp}\, P = \overline{\exp}\, \{eP + (e-1)g_0\},$$

and writing $P = g_0 + g_1\mu + g_{-1}\dfrac{1}{\mu}$, we have

$$\exp g_0 \,.\, \overline{\exp}\left(g_0 + g_1\mu + g_{-1}\frac{1}{\mu}\right) = \overline{\exp}\left\{(2e-1)g_0 + eg_1\mu + eg_{-1}\frac{1}{\mu}\right\},$$

and this leads to

$$\exp g_0 \,.\, \overline{\exp}\left(g_0 + g_1\mu + g_{-1}\frac{1}{\mu}\right) = \exp\left(\begin{aligned} & g_0 + g_1\mu - \tfrac{1}{2}g_2\mu^2 + \ldots \\ & + g_{-1}\frac{1}{\mu} - \tfrac{1}{2}g_{-2}\frac{1}{\mu^2} + \ldots \end{aligned}\right)$$

and thence to

$$\overline{\exp}\left(g_0 + g_1\mu + g_{-1}\frac{1}{\mu}\right) = \exp\left(\begin{aligned} & g_1\mu - \tfrac{1}{2}g_2\mu^2 + \tfrac{1}{3}g_3\mu^3 - \ldots \\ & + g_{-1}\frac{1}{\mu} - \tfrac{1}{2}g_{-2}\frac{1}{\mu^2} + \tfrac{1}{3}g_{-3}\frac{1}{\mu^3} - \ldots \end{aligned}\right).$$

143. Recalling previous results, we now obtain

$$\begin{aligned}1 + D_0 + D_1\mu + D_2\mu^2 + \ldots &= \exp\left\{d_0 + d_1\mu - \tfrac{1}{2}d_2\mu^2 + \ldots\right.\\ + D_{-1}\frac{1}{\mu} + D_{-2}\frac{1}{\mu^2} + \ldots &\qquad \left. + d_{-1}\frac{1}{\mu} - \tfrac{1}{2}d_{-2}\frac{1}{\mu^2} + \ldots\right\},\end{aligned}$$

$$\begin{aligned}1 + G_0 + G_1\mu + G_2\mu^2 + \ldots &= \exp\left\{g_0 + g_1\mu - \tfrac{1}{2}g_2\mu^2 + \ldots\right.\\ + G_{-1}\frac{1}{\mu} + G_{-2}\frac{1}{\mu^2} + \ldots &\qquad \left. + g_{-1}\frac{1}{\mu} - \tfrac{1}{2}g_{-2}\frac{1}{\mu^2} + \ldots\right\},\end{aligned}$$

and from these

144.

$$\begin{aligned}(1+D_0) &= e^{d_0}\left\{1 + d_1d_{-1} + \frac{d_1^2 - d_2}{2!}\cdot\frac{d_{-1}^2 - d_{-2}}{2!}\right.\\ &\qquad \left. + \frac{d_1^3 - 3d_2d_1 + 2d_3}{3!}\cdot\frac{d_{-1}^3 - 3d_{-2}d_{-1} + 2d_{-3}}{3!} + \ldots\right\},\\ D_1 &= e^{d_0}\left\{d_1 + \frac{d_1^2 - d_2}{2!}\cdot d_{-1} + \frac{d_1^3 - 3d_2d_1 + 2d_3}{3!}\cdot\frac{d_{-1}^2 - d_{-2}}{2!} + \ldots\right\},\\ D_{-1} &= e^{d_0}\left\{d_{-1} + d_1\frac{d_{-1}^2 - d_{-2}}{2!} + \frac{d_1^2 - d_2}{2!}\cdot\frac{d_{-1}^3 - 3d_{-2}d_{-1} + 2d_{-3}}{3!} + \ldots\right\},\end{aligned}$$

etc. = etc.

$$\begin{aligned}1 + G_0 &= e^{g_0}\left\{1 + g_1g_{-1} + \frac{g_1^2 - g_2}{2!}\cdot\frac{g_{-1}^2 - g_{-2}}{2!}\right.\\ &\qquad \left. + \frac{g_1^3 - 3g_2g_1 + 2g_3}{3!}\cdot\frac{g_{-1}^3 - 3g_{-2}g_{-1} + 2g_{-3}}{3!} + \ldots\right\},\\ G_1 &= e^{g_0}\left\{g_1 + \frac{g_1^2 - g_2}{2!}g_{-1} + \frac{g_1^3 - 3g_2g_1 + 2g_3}{3!}\cdot\frac{g_{-1}^2 - g_{-2}}{2!} + \ldots\right\},\\ G_{-1} &= e^{g_0}\left\{g_{-1} + g_1\frac{g_{-1}^2 - g_{-2}}{2!} + \frac{g_1^2 - g_2}{2!}\cdot\frac{g_{-1}^3 - 3g_{-2}g_{-1} + 2g_{-3}}{3!} + \ldots\right\},\end{aligned}$$

etc. = etc.

145. And also by taking logarithms

$$\begin{aligned}&d_0 + d_1\mu - \tfrac{1}{2}d_2\mu^2 + \tfrac{1}{3}d_3\mu^3 - \ldots\\ &\quad + d_{-1}\frac{1}{\mu} - \tfrac{1}{2}d_{-2}\frac{1}{\mu^2} + \tfrac{1}{3}d_{-3}\frac{1}{\mu^3} - \ldots\\ &\qquad = \log\left\{1 + D_0 + D_1\mu + D_2\mu^2 + \ldots\right.\\ &\qquad\qquad \left. + D_{-1}\frac{1}{\mu} + D_{-2}\frac{1}{\mu^2} + \ldots\right\},\end{aligned}$$

$$\begin{aligned} g_0 + g_1\mu \quad - \tfrac{1}{2} g_2\mu^2 \quad + \tfrac{1}{3} g_3\mu^3 \quad - \dots \\ + g_{-1}\frac{1}{\mu} - \tfrac{1}{2} g_{-2}\frac{1}{\mu^2} + \tfrac{1}{3} g_{-3}\frac{1}{\mu^3} - \dots \\ = \log\Big\{1 + G_0 + G_1\mu \quad + G_2\mu^2 \quad + \dots \\ + G_{-1}\frac{1}{\mu} + G_{-2}\frac{1}{\mu^2} + \dots\Big\}, \end{aligned}$$

which lead to the relations

$$g_0 = \sum \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\dots \pi_2!\,\pi_1!\;\pi_0!\;\pi_{-1}!\;\pi_{-2}!\dots} \dots G_2^{\pi_2} G_1^{\pi_1} G_0^{\pi_0} G_{-1}^{\pi_{-1}} G_{-2}^{\pi_{-2}} \dots,$$

$$\frac{(-)^{m-1}}{m} g_m \;= \text{ditto},$$

$$\frac{(-)^{m-1}}{m} g_{-m} = \text{ditto},$$

the summations having regard respectively to the solutions of the indeterminate equations

$$\Sigma t\pi_t = 0;\ \Sigma t\pi_t = m;\ \Sigma t\pi_t = -m.$$

We have also similar relations between the d and D operations, or if

$$\theta_1, \theta_2, \theta_3, \dots;\ \phi_1, \phi_2, \phi_3, \dots$$

be two sets of fictitious quantities such that

$$\begin{aligned} \left(1 + d_1x + \frac{d_1^2 - d_2}{2} + \dots\right)\left(1 + d_{-1}\frac{1}{x} + \frac{d_{-1}^2 - d_{-2}}{2}\frac{1}{x^2} + \dots\right) \\ = (1 + \theta_1x)(1 + \theta_2x)\dots\left(1 + \frac{1}{\theta_1x}\right)\left(1 + \frac{1}{\theta_2x}\right)\dots, \end{aligned}$$

$$\begin{aligned} \left(1 + g_1x + \frac{g_1^2 - g_2}{2} + \dots\right)\left(1 + g_{-1}\frac{1}{x} + \frac{g_{-1}^2 - g_{-2}}{2}\frac{1}{x^2} + \dots\right) \\ = (1 + \phi_1x)(1 + \phi_2x)\dots\left(1 + \frac{1}{\phi_1x}\right)\left(1 + \frac{1}{\phi_2x}\right)\dots, \end{aligned}$$

and we represent symmetric functions of the two sets by partitions in brackets [ ] and [ ]$'$ respectively, we have the following correspondence between quantity and operations:

I.

Quantity.

146.

$$\begin{aligned} e^n(1 + a_1x + a_2x^2 + \dots)\left(1 + a_{-1}\frac{1}{x} + a_{-2}\frac{1}{x^2} + \dots\right) \\ = e^n(1 + \alpha_1x)(1 + \alpha_2x)\dots\left(1 + \frac{1}{\alpha_1x}\right)\left(1 + \frac{1}{\alpha_2x}\right)\dots \\ = e^n\,\overline{\exp}\left\{(0) + (1)x + (\bar{1})\frac{1}{x}\right\}, \end{aligned}$$

where $\dfrac{(1)^\lambda (0)^\mu (\bar{1})^\nu}{\lambda!\ \mu!\ \nu!}$ is symbolic expression for $(1^\lambda 0^\mu \bar{1}^\nu)$

$$\begin{aligned} &= 1 + A_0 + A_1 x + A_2 x^2 + \ldots \\ &\qquad + A_{-1}\frac{1}{x} + A_{-2}\frac{1}{x^2} + \ldots \\ &= \exp\left\{\begin{aligned}&(0) + (1)x - \tfrac{1}{2}(2)x^2 + \ldots \\ &\quad + (\bar{1})\frac{1}{x} - \tfrac{1}{2}(\bar{2})\frac{1}{x^2} + \ldots\end{aligned}\right\}. \end{aligned}$$

II.

d-Operations.

147.

$$e^{d_0}\left(1 + d_1 x + \frac{d_1^2 - d_2}{2} + \ldots\right)\left(1 + d_{-1}\frac{1}{x} + \frac{d_{-1}^2 - d_{-2}}{2}\frac{1}{x^2} + \ldots\right)$$

$$= e^{d_0}(1 + \theta_1 x)(1 + \theta_2 x)\ldots\left(1 + \frac{1}{\theta_1 x}\right)\left(1 + \frac{1}{\theta_2 x}\right)\ldots$$

$$= \overline{\exp}\left\{[0] + [1]x + [\bar{1}]\frac{1}{x}\right\},$$

where $\dfrac{[1]^\lambda [0]^\mu [\bar{1}]^\nu}{\lambda!\ \mu!\ \nu!}$ is symbolic expression for $[1^\lambda 0^\mu \bar{1}^\nu]$,

$$\begin{aligned} &= 1 + D_0 + D_1 x + D_2 x^2 + \ldots \\ &\qquad + D_{-1}\frac{1}{x} + D_{-2}\frac{1}{x^2} + \ldots \\ &= \exp\left\{\begin{aligned}&[0] + [1]x - \tfrac{1}{2}[2]x^2 + \ldots \\ &\quad + [\bar{1}]\frac{1}{x} - \tfrac{1}{2}[\bar{2}]\frac{1}{x^2} + \ldots\end{aligned}\right\} \end{aligned}$$

so that $$[m] = d_m,$$

and $$[p_1^{\pi_1} p_2^{\pi_2}\ldots] = \frac{\overline{d_{p_1}^{\pi_1} d_{p_2}^{\pi_2}\ldots}}{\pi_1!\ \pi_2!\ldots}.$$

III.

g-Operations.

148.

$$e^{g_0}\left(1 + g_1 x + \frac{g_1^2 - g_2}{2} + \ldots\right)\left(1 + g_{-1}\frac{1}{x} + \frac{g_{-1}^2 - g_{-2}}{2}\frac{1}{x^2} + \ldots\right)$$

$$= e^{g_0}(1 + \phi_1 x)(1 + \phi_2 x)\ldots\left(1 + \frac{1}{\phi_1 x}\right)\left(1 + \frac{1}{\phi_2 x}\right)\ldots$$

$$= \exp[0]'\ \overline{\exp}\left\{[0]' + [1]'x + [\bar{1}]'\frac{1}{x}\right\},$$

where $\dfrac{[1]'^{\lambda}[0]'^{\mu}[\bar{1}]'^{\nu}}{\lambda!\,\mu!\,\nu!}$ is a symbolic expression for $[1^{\lambda}0^{\mu}\bar{1}^{\nu}]'$,

$$\begin{aligned} &= 1 + G_0 + G_1x + G_2x^2 + \ldots. \\ &\qquad\quad + G_{-1}\frac{1}{x} + G_{-2}\frac{1}{x^2} + \ldots. \\ &= \exp\left\{[0]' + [1]'x - \tfrac{1}{2}[2]'x^2 + \ldots.\right. \\ &\qquad\quad \left. + [\bar{1}]'\frac{1}{x} - \tfrac{1}{2}[\bar{2}]'\frac{1}{x^2} + \ldots.\right\}, \end{aligned}$$

so that

$$[m]' = g_m,$$

$$[p_1^{\pi_1}p_2^{\pi_2}]' = \frac{g_{p_1}^{\pi_1}g_{p_2}^{\pi_2}\ldots.}{\pi_1!\,\pi_2!\ldots.}.$$

§10.

149. I now apply the foregoing section to a new demonstration of the "Law of Reciprocity" in the theory of separations of which a purely arithmetical proof was given in the second memoir.

Consider three identities

$$\begin{aligned} &1 + A_0 + A_1x + A_2x^2 + \ldots. + A_{-1}\frac{1}{x} + A_{-2}\frac{1}{x^2} + \ldots., \\ &\qquad = e^{n}(1+\alpha_1x)(1+\alpha_2x)\ldots.\left(1+\frac{1}{\alpha_1x}\right)\left(1+\frac{1}{\alpha_2x}\right)\ldots. \\ &1 + B_0 + B_1x + B_2x^2 + \ldots. + B_{-1}\frac{1}{x} + B_{-2}\frac{1}{x^2} + \ldots., \\ &\qquad = e^{n'}(1+\beta_1x)(1+\beta_2x)\ldots.\left(1+\frac{1}{\beta_1x}\right)\left(1+\frac{1}{\beta_2x}\right)\ldots., \\ &1 + C_0 + C_1x + C_2x^2 + \ldots. + C_{-1}\frac{1}{x} + C_{-2}\frac{1}{x^2} + \ldots., \\ &\qquad = e^{n''}(1+\gamma_1x)(1+\gamma_2x)\ldots.\left(1+\frac{1}{\gamma_1x}\right)\left(1+\frac{1}{\gamma_2x}\right)^{*}\ldots., \end{aligned}$$

and let symmetric functions of the quantities

$\alpha_1, \alpha_2, \alpha_3, \ldots.$ be denoted by partitions in $(\ )_\alpha$,
$\beta_1, \beta_2, \beta_3, \ldots.$ " " " " " $(\ )_\beta$,
$\gamma_1, \gamma_2, \gamma_3, \ldots.$ " " " " " $(\ )_\gamma$,

---

* n, n', n'' are each to be supposed indefinitely great, and further $n = n'$.

then we have a triad of identities.

150.

$$\left.\begin{array}{l}1+A_0+A_1x\quad+A_2x^2\quad+\ldots=\exp\{(0)_\alpha+(1)_\alpha x\quad-\tfrac{1}{2}(2)_\alpha x^2\quad+\ldots\\ \quad+A_{-1}\dfrac{1}{x}+A_{-2}\dfrac{1}{x^2}+\ldots\qquad\qquad+(\bar{1})_\alpha\dfrac{1}{x}-\tfrac{1}{2}(\bar{2})_\alpha\dfrac{1}{x^2}+\ldots\end{array}\right\},$$

$$\left.\begin{array}{l}1+B_0+B_1x\quad+B_2x^2\quad+\ldots=\exp\{(0)_\beta+(1)_\beta x\quad-\tfrac{1}{2}(2)_\beta x^2\quad+\ldots\\ \quad+B_{-1}\dfrac{1}{x}+B_{-2}\dfrac{1}{x^2}+\ldots\qquad\qquad+(\bar{1})_\beta\dfrac{1}{x}-\tfrac{1}{2}(\bar{2})_\beta\dfrac{1}{x^2}+\ldots\end{array}\right\},$$

$$\left.\begin{array}{l}1+C_0+C_1x\quad+C_2x^2\quad+\ldots=\exp\{(0)_\gamma+(1)_\gamma x\quad-\tfrac{1}{2}(2)_\gamma x^2\quad+\ldots\\ \quad+C_{-1}\dfrac{1}{x}+C_{-2}\dfrac{1}{x^2}+\ldots\qquad\qquad+(\bar{1})_\gamma\dfrac{1}{x}-\tfrac{1}{2}(\bar{2})_\gamma\dfrac{1}{x^2}+\ldots\end{array}\right\}.$$

151. Now assume that, between the quantities herein involved, there exists the relation

$$\begin{aligned}1+C_0+C_1y\quad&+C_2y^2\quad+\ldots.\\ +C_{-1}\frac{1}{y}&+C_{-2}\frac{1}{y^2}+\ldots.\\ &=\prod_{s=1}^{s=n}\left(1+B_0+\alpha_sB_1y\qquad+\alpha_s^2B_2y^2\qquad+\ldots.\right.\\ &\qquad\qquad\left.+\frac{1}{\alpha_s}B_{-1}\frac{1}{y}+\frac{1}{\alpha_s^2}B_{-2}\frac{1}{y^2}+\ldots.\right),\end{aligned}$$

a relation which also implies the identity

$$1+c_1y+c_2y^2+c_3y^3+\ldots.=\prod_{s=1}^{s=n}(1+\alpha_sb_1y+\alpha_s^2b_2y^2+\alpha_s^3b_3y^3+\ldots.),$$

y being arbitrary, so that

$$\begin{aligned}\log\left(1+C_0+C_1y\quad+C_2y^2\quad+\ldots.\right.\\ \left.+C_{-1}\frac{1}{y}+C_{-2}\frac{1}{y^2}+\ldots.\right)\\ =\sum_{s=1}^{s=n}\log\left(1+B_0+\alpha_sB_1y\qquad+\alpha_s^2B_2y^2\qquad+\ldots.\right.\\ \left.+\frac{1}{\alpha_s}B_{-1}\frac{1}{y}+\frac{1}{\alpha_s^2}B_{-2}\frac{1}{y^2}+\ldots.\right),\end{aligned}$$

leading to

$$\begin{array}{l}(0)_\gamma+(1)_\gamma y\quad-\tfrac{1}{2}(2)_\gamma y^2\quad+\ldots.=(0)_\alpha(0)_\beta+(1)_\alpha(1)_\beta y\quad-\tfrac{1}{2}(2)_\alpha(2)_\beta y^2\quad+\ldots.\\ \quad+(\bar{1})_\gamma\dfrac{1}{y}-\tfrac{1}{2}(\bar{2})_\gamma\dfrac{1}{y^2}+\ldots.\qquad\qquad+(\bar{1})_\alpha(\bar{1})_\beta\dfrac{1}{y}-\tfrac{1}{2}(\bar{2})_\alpha(\bar{2})_\beta\dfrac{1}{y^2}+\ldots.\end{array}$$

and thence to $$(m)_\gamma=(m)_\alpha(m)_\beta,$$

where m is any integer, positive, zero, or negative.

152. This result, which is of great importance, shows that the function of $C_0, C_1, C_{-1}, C_2, C_{-2}, \ldots$, denoted by $(m)_\gamma$, is unaltered when the n quantities $\alpha_1, \alpha_2, \alpha_3, \ldots$ and the several n' quantities $\beta_1, \beta_2, \beta_3, \ldots$ are interchanged; but every symmetric function is expressible in terms of sums of powers of the quantities, and it hence follows that every symmetric function of the n quantities

$$\gamma_1, \gamma_2, \gamma_3, \ldots$$

remains unaltered by the interchange of the n quantities

$$\alpha_1, \alpha_2, \alpha_3, \ldots$$

with the several n' quantities

$$\beta_1, \beta_2, \beta_3, \ldots;$$

we may say, in fact, that if any assemblage of partitions in brackets

$$(\quad)_\gamma$$

be expressed in terms of partitions in brackets

$$(\quad)_\alpha \text{ and } (\quad)_\beta,$$

it remains unaltered by the interchange of the brackets $(\quad)_\alpha$ and $(\quad)_\beta$.

153. For example, it is shown in this way that if we have a result

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots)_\gamma = \ldots + J(p_1^{\pi_1} p_2^{\pi_2} \ldots)_\alpha (\lambda_1^{l_1} \lambda_2^{l_2} \ldots)_\beta + \ldots$$

we must also have

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots)_\gamma = \ldots + J\{(p_1^{\pi_1} p_2^{\pi_2} \ldots)_\alpha (\lambda_1^{l_1} \lambda_2^{l_2} \ldots)_\beta + (p_1^{\pi_1} p_2^{\pi_2} \ldots)_\beta (\lambda_1^{l_1} \lambda_2^{l_2} \ldots)_\alpha\} + \ldots,$$

and this fact will be shown to involve the "Law of Reciprocity" brought forward in the first and second memoirs.

154. It should be remarked that, as a consequence, the assumed relation may be also written in the form

$$\begin{aligned} 1 + C_0 + C_1 y + C_2 y^2 + \ldots & \\ + C_{-1}\frac{1}{y} + C_{-2}\frac{1}{y^2} + \ldots & \\ = \prod_{s=1}^{s=n} \Big(1 + A_0 + \beta_s A_1 y & + \beta_s^2 A_2 y^2 + \ldots \\ + \frac{1}{\beta_s} A_{-1}\frac{1}{y} & + \frac{1}{\beta_s^2} A_{-2}\frac{1}{y^2} + \ldots \Big). \end{aligned}$$

155. Associated with the triad of identities above set forth, we have a triad of operator relations, which I will write as follows:

$$\begin{aligned}1 + {}_{\alpha}G_0 + {}_{\alpha}G_1 y + {}_{\alpha}G_2 y^2 + \ldots\\ + {}_{\alpha}G_{-1}\frac{1}{y} + {}_{\alpha}G_{-2}\frac{1}{y^2} + \ldots\\ = \exp\left\{\begin{aligned}[0]'_{\alpha} + [1]'_{\alpha} y - \tfrac{1}{2}[2]'_{\alpha} y^2 + \ldots\\ + [\overline{1}]'_{\alpha}\frac{1}{y} - \tfrac{1}{2}[\overline{2}]'_{\alpha}\frac{1}{y^2} + \ldots\end{aligned}\right\},\end{aligned}$$

where $$[m]'_{\alpha} = {}_{\alpha}g_m,$$

$$[p_1^{\pi_1}p_2^{\pi_2}\ldots]'_{\alpha} = \frac{\overline{{}_{\alpha}g_{p_1}^{\pi_1}\,{}_{\alpha}g_{p_2}^{\pi_2}\ldots}}{\pi_1!\,\pi_2!\ldots},$$

$$\begin{aligned}1 + {}_{\beta}G_0 + {}_{\beta}G_1 y + {}_{\beta}G_2 y^2 + \ldots\\ + {}_{\beta}G_{-1}\frac{1}{y} + {}_{\beta}G_{-2}\frac{1}{y^2} + \ldots\\ = \exp\left\{\begin{aligned}[0]'_{\beta} + [1]'_{\beta} y - \tfrac{1}{2}[2]'_{\beta} y^2 + \ldots\\ + [\overline{1}]_{\beta}\frac{1}{y} - \tfrac{1}{2}[\overline{2}]'_{\beta}\frac{1}{y^2} + \ldots\end{aligned}\right\},\end{aligned}$$

where $$[m]'_{\beta} = {}_{\beta}g_m,$$

$$[p_1^{\pi_1}p_2^{\pi_2}\ldots]'_{\beta} = \frac{\overline{{}_{\beta}g_{p_1}^{\pi_1}\,{}_{\beta}g_{p_2}^{\pi_2}\ldots}}{\pi_1!\,\pi_2!\ldots},$$

$$\begin{aligned}1 + {}_{\gamma}G_0 + {}_{\gamma}G_1 y + {}_{\gamma}G_2 y^2 + \ldots\\ + {}_{\gamma}G_{-1}\frac{1}{y} + {}_{\gamma}G_{-2}\frac{1}{y^2} + \ldots\\ = \exp\left\{\begin{aligned}[0]'_{\gamma} + [1]'_{\gamma} y - \tfrac{1}{2}[2]'_{\gamma} y^2 + \ldots\\ + [\overline{1}]'_{\gamma}\frac{1}{y} - \tfrac{1}{2}[\overline{2}]'_{\gamma}\frac{1}{y^2} + \ldots\end{aligned}\right\},\end{aligned}$$

where $$[m]'_{\gamma} = {}_{\gamma}g_m,$$

$$[p_1^{\pi_1}p_2^{\pi_2}\ldots]'_{\gamma} = \frac{\overline{{}_{\gamma}g_{p_1}^{\pi_1}\,{}_{\gamma}g_{p_2}^{\pi_2}\ldots}}{\pi_1!\,\pi_2!\ldots}.$$

156. Now writing the assumed relation, viz.

$$\begin{aligned}1 + C_0 + C_1 y + C_2 y^2 + \ldots\\ + C_{-1}\frac{1}{y} + C_{-2}\frac{1}{y^2} + \ldots\\ = \prod_{s=1}^{s=n}\left(\begin{aligned}1 + B_0 + \alpha_s B_1 y + \alpha_s^2 B_2 y^2 + \ldots\\ + \frac{1}{\alpha_s}B_{-1}\frac{1}{y} + \frac{1}{\alpha_s^2}B_{-2}\frac{1}{y^2} + \ldots\end{aligned}\right)\end{aligned}$$

in the abbreviated form

$$U = u_{\alpha_1} u_{\alpha_2} u_{\alpha_3} \ldots,$$

we have

$$_{\beta}g_m U = ({}_{\beta}g_m u_{\alpha_1}) u_{\alpha_2} u_{\alpha_3} \ldots + u_{\alpha_1} ({}_{\beta}g_m u_{\alpha_2}) u_{\alpha_3} \ldots + u_{\alpha_1} u_{\alpha_2} ({}_{\beta}g_m u_{\alpha_3}) \ldots + \ldots$$

and

$$\begin{aligned} {}_{\beta}g_m u_{\alpha_1} &= \left\{ \begin{matrix} (1+B_0)\partial_{B_m} + B_1 \partial_{B_{m+1}} + \ldots \\ + B_{-1}\partial_{B_{m-1}} + \ldots \end{matrix} \right\} \left(\begin{matrix} 1 + B_0 + \alpha_1 B_1 y + \alpha_1^2 B_2 y^2 + \ldots \\ + \frac{1}{\alpha_1} B_{-1} \frac{1}{y} + \frac{1}{\alpha_1^2} B_{-2} \frac{1}{y^2} + \ldots \end{matrix} \right) \\ &= \alpha_1^m y^m u_{\alpha_1}; \end{aligned}$$

hence $\qquad {}_{\beta}g_m U = (\alpha_1^m + \alpha_2^m + \alpha_3^m + \ldots) y^m U = (m)_\alpha y^m U,$

leading to

$$\begin{aligned} &{}_{\beta}g_m (1 + C_0) + y\, {}_{\beta}g_m C_1 + y^2\, {}_{\beta}g_m C_2 + \ldots \\ &\qquad + \frac{1}{y}\, {}_{\beta}g_m C_{-1} + \frac{1}{y^2}\, {}_{\beta}g_m C_{-2} + \ldots \\ &\qquad\qquad = (m)_\alpha y^m \left\{ \begin{matrix} 1 + C_0 + C_1 y + C_2 y^2 + \ldots \\ + C_{-1} \frac{1}{y} + C_{-2} \frac{1}{y^2} + \ldots \end{matrix} \right\}, \end{aligned}$$

which gives, on equating coefficients of like powers of y,

$$\begin{aligned} {}_{\beta}g_m (1 + C_0) &= (m)_\alpha C_{-m}, \\ {}_{\beta}g_m C_1 &= (m)_\alpha C_{-m+1}, \\ {}_{\beta}g_m C_{-1} &= (m)_\alpha C_{-m-1}, \\ &\ldots\ldots\ldots\ldots \\ {}_{\beta}g_m C_m &= (m)_\alpha (1 + C_0); \end{aligned}$$

but regarding $B_0, B_1, B_{-1}, B_2, B_{-2}, \ldots$ as functions of $C_0, C_1, C_{-1}, C_2, C_{-2}, \ldots$ only, we have

$$\begin{aligned} {}_{\beta}g_m &\equiv {}_{\beta}g_m (1 + C_0) \partial_{C_0} + {}_{\beta}g_m C_1 \partial_{C_1} + {}_{\beta}g_m C_2 \partial_{C_2} + \ldots \\ &\qquad + {}_{\beta}g_m C_{-1} \partial_{C_{-1}} + {}_{\beta}g_m C_{-2} \partial_{C_{-2}} + \ldots \\ &= (m)_\alpha \left\{ \begin{matrix} C_{-m} \partial_{C_0} + C_{-m+1} \partial_{C_1} + C_{-m+2} \partial_{C_2} + \ldots \\ + C_{-m-1} \partial_{C_{-1}} + C_{-m-2} \partial_{C_{-2}} + \ldots \end{matrix} \right\}, \end{aligned}$$

or $\qquad {}_{\beta}g_m = (m)_\alpha\, {}_{\gamma}g_m;$

we thus arrive at the conclusion that, regarding the assumed relation as defining a transformation of any function of $C_0, C_1, C_{-1}, C_2, C_{-2}, \ldots$ into a function

of $B_0, B_1, B_{-1}, B_2, B_{-2}, \ldots, \alpha_1, \alpha_2, \alpha_3, \ldots$ being the constants of the transformation, the operation

$$_{\gamma}g_m$$

is an invariant.

157. It may be remarked by the way that

$$_{\gamma}g_m \log\left\{\begin{array}{l} 1 + C_0 + C_1 x \quad + C_2 x^2 \quad + \ldots \\ \qquad\qquad + C_{-1}\dfrac{1}{x} + C_{-2}\dfrac{1}{x^2} + \ldots \end{array}\right\} = x^m,$$

whence

$$_{\gamma}g_m \left\{\begin{array}{l} (0)_\gamma + (1)_\gamma x \quad - \tfrac{1}{2}(2)_\gamma x^2 \quad + \ldots \\ \qquad\quad + (\overline{1})_\gamma \dfrac{1}{x} - \tfrac{1}{2}(\overline{2})\dfrac{1}{x^2} + \ldots \end{array}\right\} = x^m,$$

that is,

$$_{\gamma}g_m \frac{(-)^{m+1}}{m}(m)_\gamma = 1,$$

$$_{\gamma}g_{-m} \frac{(-)^{m+1}}{m}(\overline{m})_\gamma = 1,$$

where m differs from zero, and

$$_{\gamma}g_0 (0)_\gamma = 1,$$

whilst

$$_{\gamma}g_m (s)_\gamma = 0 \text{ if } s \neq m;$$

we have thus a set of transcendental solutions of the partial differential equation

$$_{\gamma}g_m = 0,$$

and all of these solutions have otherwise been proved to be invariants of the supposed transformation (vide second memoir).

158. Returning to the relation

$$_{\beta}g_m = (m)_\alpha\, {}_{\gamma}g_m,$$

we may write

$$\begin{array}{l} _{\beta}g_0 + {}_{\beta}g_1 y \quad - \tfrac{1}{2}\,{}_{\beta}g_2 y^2 \quad + \ldots \\ \quad + {}_{\beta}g_{-1}\dfrac{1}{y} - \tfrac{1}{2}\,{}_{\beta}g_{-2}\dfrac{1}{y^2} + \ldots \end{array} = \begin{array}{l} (0)_\alpha + (1)_\alpha\, {}_{\gamma}g_1 y \quad - \tfrac{1}{2}(2)_\alpha\, {}_{\gamma}g_2 y^2 \quad + \ldots \\ \quad + (\overline{1})_\alpha\, {}_{\gamma}g_{-1}\dfrac{1}{y} - \tfrac{1}{2}(\overline{2})_\alpha\, {}_{\gamma}g_{-2}\dfrac{1}{y^2} + \ldots, \end{array}$$

y being arbitrary, and also in the form

$$\exp\left\{\begin{array}{l} [0]'_\beta + [1]'_\beta y \quad - \tfrac{1}{2}[2]'_\beta y^2 \quad + \ldots \\ \qquad + [\overline{1}]'_\beta \dfrac{1}{y} - \tfrac{1}{2}[\overline{2}]'_\beta \dfrac{1}{y^2} + \ldots \end{array}\right\},$$

$$= \exp\left\{\begin{array}{l} (0)_\alpha[0]'_\gamma + (1)_\alpha[1]'_\gamma y \quad - \tfrac{1}{2}(2)_\alpha[2]'_\gamma y^2 \quad + \ldots \\ \qquad + (\overline{1})_\alpha[\overline{1}]'_\gamma \dfrac{1}{y} - \tfrac{1}{2}(\overline{2})_\alpha[\overline{2}]'_\gamma \dfrac{1}{y^2} + \ldots \end{array}\right\},$$

and then from previous work we are led to the result

159.

$$\begin{aligned}1 + {}_\beta G_0 + {}_\beta G_1 y + {}_\beta G_2 y^2 + \dots\\ + {}_\beta G_{-1}\frac{1}{y} + {}_\beta G_{-2}\frac{1}{y^2} + \dots\\ = \prod_{s=1}^{s=n}\Big(1 + {}_\gamma G_0 + \alpha_s {}_\gamma G_1 y + \alpha_s^2 {}_\gamma G_2 y^2 + \dots\\ + \frac{1}{\alpha_s} {}_\gamma G_{-1}\frac{1}{y} + \frac{1}{\alpha_s^2} {}_\gamma G_{-2}\frac{1}{y^2} + \dots\Big),\end{aligned}$$

and a comparison with the assumed relation

$$\begin{aligned}1 + C_0 + C_1 y + C_2 y^2 + \dots\\ + C_{-1}\frac{1}{y} + C_{-2}\frac{1}{y^2} + \dots\\ = \prod_{s=1}^{s=n}\Big(1 + B_0 + \alpha_s B_1 y + \alpha_s^2 B_2 y^2 + \dots\\ + \frac{1}{\alpha_s} B_{-1}\frac{1}{y} + \frac{1}{\alpha_s^2} B_{-2}\frac{1}{y^2} + \dots\Big)\end{aligned}$$

leads to the following theorem:

160. "In any relation connecting the quantities

$$C_0,\ C_1,\ C_{-1},\ C_2,\ C_{-2},\ \dots$$

with the quantities

$$B_0,\ B_1,\ B_{-1},\ B_2,\ B_{-2},\ \dots$$

we are at liberty to substitute

$${}_\beta G_\kappa \text{ for } C_\kappa$$

and

$${}_\gamma G_\kappa \text{ for } B_\kappa,$$

and we so obtain a relation between operators."

This very important theorem can be applied forthwith.

161. By means of the initial relation

$$\begin{aligned}1 + C_0 + C_1 x + C_2 x^2 + \dots\\ + C_{-1}\frac{1}{x} + C_{-2}\frac{1}{x^2} + \dots\\ = \prod_{s=1}^{s=n}\Big(1 + B_0 + \alpha_s B_1 x + \alpha_s^2 B_2 x^2 + \dots\\ + \frac{1}{\alpha_s} B_{-1}\frac{1}{x} + \frac{1}{\alpha_s^2} B_{-2}\frac{1}{x^2} + \dots\Big).\end{aligned}$$

Where x is arbitrary,* we can express $C_0, C_1, C_{-1}, C_2, C_{-2}, \ldots$ in terms of the quantities $B_0, B_1, B_{-1}, B_2, B_{-2}, \ldots$ and monomial symmetric functions of the n quantities

$$\alpha_1, \alpha_2, \ldots \alpha_n;$$

and, multiplying out, we obtain a result such as

$$C_{p_1}^{\pi_1} C_{p_2}^{\pi_2} \ldots = \ldots + L B_{s_1}^{\sigma_1} B_{s_2}^{\sigma_2} \ldots + \ldots, \tag{I}$$

and also a result such as

$$C_{\lambda_1}^{l_1} C_{\lambda_2}^{l_2} \ldots = \ldots + M B_{s_1}^{\sigma_1} B_{s_2}^{\sigma_2} \ldots + \ldots \tag{II}$$

Join to these two, a third, viz.

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots)_\gamma = \ldots + A\,(p_1^{\pi_1} p_2^{\pi_2} \ldots)_\beta + B\,(\lambda_1^{l_1} \lambda_2^{l_2} \ldots)_\beta + \ldots \tag{III}$$

Now the equation (I) yields the equation of operators†

$$_\beta G_{p_1}^{\pi_1}\, {}_\beta G_{p_2}^{\pi_2} \ldots = \ldots + L\, {}_\gamma G_{s_1}^{\sigma_1}\, {}_\gamma G_{s_2}^{\sigma_2} \ldots + \ldots,$$

and performing each side of this upon the opposite side of the relation (III), we obtain

$$L\, {}_\gamma G_{s_1}^{\sigma_1}\, {}_\gamma G_{s_2}^{\sigma_2} \ldots (s_1^{\sigma_1} s_2^{\sigma_2} \ldots)_\gamma = A\, {}_\beta G_{p_1}^{\pi_1}\, {}_\beta G_{p_2}^{\pi_2} \ldots (p_1^{\pi_1} p_2^{\pi_2} \ldots)_\beta,$$

no other terms surviving the operation; but

$$_\gamma G_{s_1}^{\sigma_1}\, {}_\gamma G_{s_2}^{\sigma_2} \ldots (s_1^{\sigma_1} s_2^{\sigma_2} \ldots)_\gamma = {}_\beta G_{p_1}^{\pi_1}\, {}_\beta G_{p_2}^{\pi_2} \ldots (p_1^{\pi_1} p_2^{\pi_2} \ldots)_\beta = 1;$$

hence $$L = A,$$

and also from the identities (II) and (III) we obtain

$$M = B,$$

and this leads to

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots)_\gamma = \ldots + L\,(p_1^{\pi_1} p_2^{\pi_2} \ldots)_\beta + M\,(\lambda_1^{l_1} \lambda_2^{l_2} \ldots)_\beta + \ldots$$

Now, it has been shown previously that

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots)_\gamma = \ldots + J\,\{(\lambda_1^{l_1} \lambda_2^{l_2} \ldots)_\alpha (p_1^{\pi_1} p_2^{\pi_2} \ldots)_\beta + (\lambda_1^{l_1} \lambda_2^{l_2} \ldots)_\beta (p_1^{\pi_1} p_2^{\pi_2} \ldots)_\alpha\} + \ldots;$$

---

* This relation takes the place of the relation in the second memoir, viz.

$$\begin{aligned} 1 + X_0 + X_1 x + X_2 x^2 + \ldots &= \prod_{s=1}^{s=n} \Big(1 + x_0 + a_s x_1 x + a_s^2 x_2 x^2 + \ldots \\ + X_{-1}\frac{1}{x} + X_{-2}\frac{1}{x^2} + \ldots &\qquad + \frac{1}{a_s} x_{-1} \frac{1}{x} + \frac{1}{a_s^2} x_{-2} \frac{1}{x^2} + \ldots\Big), \end{aligned}$$

the notation alone being changed.

† Compare Hammond, loc. cit.

hence we must have

$$L = \ldots\ldots + J(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots)_a + \ldots\ldots,$$
$$M = \ldots\ldots + J(p_1^{\pi_1}p_2^{\pi_2}\ldots\ldots)_a + \ldots\ldots,$$

162. And then the relations (I) and (II) become

$$C_{p_1}^{\pi_1}C_{p_2}^{\pi_2}\ldots\ldots = \ldots\ldots + J(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots)_a B_{s_1}^{\sigma_1}B_{s_2}^{\sigma_2}\ldots\ldots + \ldots\ldots,$$
$$C_{\lambda_1}^{l_1}C_{\lambda_2}^{l_2}\ldots\ldots = \ldots\ldots + J(p_1^{\pi_1}p_2^{\pi_2}\ldots\ldots)_a B_{s_1}^{\sigma_1}B_{s_2}^{\sigma_2}\ldots\ldots + \ldots\ldots,$$

which is the law of reciprocity it was required to establish.

§11.

New Law of Symmetry.

163. Suppose that we find the relation

$$(p_1^{\pi_1}p_2^{\pi_2}\ldots\ldots)_\gamma = \ldots\ldots + PB_{s_1}^{\sigma_1}B_{s_2}^{\sigma_2}\ldots\ldots + \ldots\ldots,$$

leading to the operator relation

$$\frac{\overline{{}_\beta g_{p_1}^{\pi_1}\,{}_\beta g_{p_2}^{\pi_2}\ldots\ldots}}{\pi_1!\;\pi_2!\ldots\ldots} = \ldots\ldots + P\,{}_\gamma G_{s_1}^{\sigma_1}\,{}_\gamma G_{s_2}^{\sigma_2}\ldots\ldots + \ldots\ldots,$$

where P is an aggregate of symmetric functions of the quantities $\alpha_1, \alpha_2, \ldots \alpha_n$.

Further, suppose that

$$(s_1^{\sigma_1}s_2^{\sigma_2}\ldots\ldots)_\gamma = \ldots\ldots + QB_{p_1}^{\pi_1}B_{p_2}^{\pi_2}\ldots\ldots + \ldots\ldots,$$

since
$$_\beta g_m = \partial_{B_m} + B_0\partial_{B_m} + B_1\partial_{B_{m+1}} + \ldots\ldots$$
$$+ B_{-1}\partial_{B_{m-1}} + \ldots\ldots,$$

we have $\dfrac{\overline{{}_\beta g_{p_1}^{\pi_1}\,{}_\beta g_{p_2}^{\pi_2}\ldots\ldots}}{\pi_1!\;\pi_2!\ldots\ldots} = \dfrac{\partial_{B_{p_1}}^{\pi_1}\partial_{B_{p_2}}^{\pi_2}\ldots\ldots}{\pi_1!\;\pi_2!\ldots\ldots}$ + terms which, operating upon a function of $B_0, B_1, B_{-1}, \ldots\ldots$, do not diminish its degree; hence, attending only to terms of like weight and degree,

$$P\,{}_\gamma G_{s_1}^{\sigma_1}\,{}_\gamma G_{s_2}^{\sigma_2}\ldots\ldots(s_1^{\sigma_1}s_2^{\sigma_2}\ldots\ldots)_\gamma = Q\frac{\partial_{B_{p_1}}^{\pi_1}\partial_{B_{p_2}}^{\pi_2}\ldots\ldots}{\pi_1!\;\pi_2!\ldots\ldots}B_{p_1}^{\pi_1}B_{p_2}^{\pi_2}\ldots\ldots,$$

that is, $P = Q$; therefore we have the theorem

164. "If

$$(p_1^{\pi_1}p_2^{\pi_2}\ldots\ldots)_\gamma = \ldots\ldots + A(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots)_a B_{s_1}^{\sigma_1}B_{s_2}^{\sigma_2}\ldots\ldots + \ldots\ldots,$$

then
$$(s_1^{\sigma_1}s_2^{\sigma_2}\ldots\ldots)_\gamma = \ldots\ldots + A(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots)_a B_{p_1}^{\pi_1}B_{p_2}^{\pi_2}\ldots\ldots + \ldots\ldots."$$

165. To make manifest the importance of this theorem I take a concrete case and, for simplicity, restrict myself to symmetric functions which are expressible by means of partitions composed merely of positive integers. We have now, as arguments, the quantities

$$b_1, b_2, b_3, \ldots . ; \; c_1, c_2, c_3, \ldots .$$

derived from the relations

$$1 + b_1x + b_2x^2 + \ldots . = (1 + \beta_1x)(1 + \beta_2x)(1 + \beta_3x) \ldots . ,$$
$$1 + c_1x + c_2x^2 + \ldots . = (1 + \gamma_1x)(1 + \gamma_2x)(1 + \gamma_3x) \ldots . ,$$

and then the fundamental relation leads to the set of identities

$$c_1 = (1)_a b_1,$$
$$c_2 = (2)_a b_2 + (1^2)_a b_1^2,$$
$$c_3 = (3)_a b_3 + (21)_a b_2 b_1 + (1^3)_a b_1^3,$$
$$c_4 = (4)_a b_4 + (31)_a b_3 b_1 + (2^2)_a b_2^2 + (21^2)_a b_2 b_1^2 + (1^4)_a b_1^4,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and we can, for example, form a table of the fourth order by expressing the symmetric functions of $\gamma_1, \gamma_2, \gamma_3, \ldots .$ of weight four in terms of the quantities $b_1, b_2, b_3, b_4, \ldots .$

166. Such a table is now given:

	b_4	b_3b_1	b_2^2	$b_2b_1^2$	b_1^4
$(4)_\gamma$	$-4(4)$	$4(3)(1)-4(31)$	$2(2)^2-4(2^2)$	$-4(2)(1)^2+4(2)(1^2)$ $+4(21)(1)-4(21^2)$	$(1)^4-4(1^2)(1)^2+2(1^2)^2$ $+4(1^3)(1)-4(1^4)$
$(31)_\gamma$	$4(4)$	$-(3)(1)+4(31)$	$-2(2)^2+4(2^2)$	$(2)(1)^2-(21)(1)$ $-4(2)(1^2)+4(21^2)$	$(1)^2(1^2)-(1^3)(1)$ $-2(1^2)^2+4(1^4)$
$(2^2)_\gamma$	$2(4)$	$-2(3)(1)+2(31)$	$(2)^2+2(2^2)$	$2(2)(1^2)-2(21)(1)$ $+2(21^2)$	$(1^2)^2-2(1^3)(1)$ $+2(1^4)$
$(21^2)_\gamma$	$-4(4)$	$(3)(1)-4(31)$	$-4(2^2)$	$(21)(1)-4(21^2)$	$(1^3)(1)-4(1^4)$
$(1^4)_\gamma$	(4)	(31)	(2^2)	(21^2)	(1^4)

This is to be read from left to right; for instance, the last line is read

$$(1^4)_\gamma = (4)\,b_4 + (31)\,b_3b_1 + (2^2)\,b_2^2 + (21^2)\,b_2b_1^2 + (1^4)\,b_1^4.$$

167. In this table the suffix α has been for convenience omitted.

The theorem shows that the s^{th} row and the s^{th} column are identical; the table in fact possesses, what is termed, row and column symmetry.

In any column the terms are all separations of the partition of the B-product at the head of the column.

In any row the assemblages of separations are formed according to a law defined by the γ partition at the left of the row.

168. To explain this I form the assemblage of separations of (21^2) according to the law defined by the partition (31); the process consists in first writing down the expression of (31) in terms of separations of (1^4), thus:

$$(31) = (1^2)(1)^2 - 2(1^2)^2 - (1^3)(1) + 4(1^4),$$

the specification of each term is then written down and, beneath, the corresponding coefficient, thus:

$$\left.\begin{array}{lcccc} \text{Specifications, } \ldots\ldots & (21^2) & (2^2) & (31) & (4) \\ \text{Coefficients, } \ldots\ldots & +1 & -2 & -1 & +4 \end{array}\right\}.$$

The two lines, last written, define the law which is applied to the case in hand as follows: The line of specifications is again written down, and underneath each specification those separations of (21^2) which are of that specification, care being taken to write down a separation in correspondence with each permutation amongst separates of the same weight.

We thus obtain two lines, viz.

Specifications.	(21^2)	(2^2)	(31)	(4)
Separations.	$(2)(1)^2$	$(2)(1^2)$ $(1^2)(2)$	$(21)(1)$	(21^2)

We finally attach the coefficients, which appear in the definition of the law, to the separations of corresponding specification; the result is the assemblage

$$(2)(1)^2 - 2(2)(1^2) - (21)(1) + 4(21^2)$$
$$- 2(1^2)(2),$$

or

$$(2)(1)^2 - 4(2)(1^2) - (21)(1) + 4(21^2),$$

which will be found in the second row and fourth column of the above table.

169. The process is conveniently placed in four rows as follows:

(31)	Specifications,	(21^2)	(2^2)	(31)	(4)
	Coefficients,	$+1$	-2	-1	$+4$
(21^2)	Separations,	$(2)(1)^2$	$(2)(1^2)$ $(1^2)(2)$	$(21)(1)$	(21^2)
	Assemblage,	$(2)(1)^2$	$-4(2)(1^2)$	$-(21)(1)$	$+4(21^2)$

Now, observe that the assemblage which occurs in the fourth row and second column is identical with that just found, and moreover is formed from separations of (31) according to the law defined by the partition (21^2), since

$$(21^2) = (1^3)(1) - 4(1^4).$$

The process is as under:

(21^2)	Specifications,	(31)	(4)
	Coefficients,	$+1$	-4
(31)	Separations,	$(3)(1)$	(31)
	Assemblage,	$(3)(1)$	$-4(31)$

and the assemblage thus found is necessarily in the fourth row and second column of the table.

170. I now enunciate a law of symmetry.

Theorem:

"If any symmetric function $(p_1^{\pi_1}p_2^{\pi_2}\ldots.)$ of weight n be expressed in terms of separations of the symmetric function (1^n), a term of specification $(\lambda_1^{l_1}\lambda_2^{l_2}\ldots.)$ is found attached to a certain numerical coefficient x; two rows of specifications

and corresponding numerical coefficients respectively are said to define the 'law of the symmetric function $(p_1^{\pi_1} p_2^{\pi_2} \dots .)$.' The assemblage of separations of a symmetric function $(s_1^{\sigma_1} s_2^{\sigma_2} \dots .)$, formed according to the law of the symmetric function $(p_1^{\pi_1} p_2^{\pi_2} \dots .)$ is identical with the assemblage, of separations of the symmetric function $(p_1^{\pi_1} p_2^{\pi_2} \dots .)$, formed according to the law of the symmetric function $(s_1^{\sigma_1} s_2^{\sigma_2} \dots .)$."

171. I subjoin an additional example of the theorem, indicating the reciprocity between the functions

(31^3) and $(2^2 1^2)$ of weight 6.

$$(2^2 1^2) = (1^4)(1^2) \quad - 4(1^5)(1) \quad + 9(1^6),$$

$(2^2 1^2)$	Specifications,	(42)	(51)	(6)
	Coefficients,	$+1$	-4	$+9$
(31^3)	Separations,	$(31)(1^2)$	$(31^2)(1)$	(31^3)
	Assemblage,	$(31)(1^2)$	$-4(31^2)(1)$	$+9(31^3)$

$$(31^3) = (1^4)(1)^2 \quad - 2(1^4)(1^2) \quad - (1^5)(1) \quad + 6(1^6)$$

(31^3)	Specifications,	(41^2)	(42)	(51)	(6)
	Coefficients,	$+1$	-2	-1	$+6$
$(2^2 1^2)$	Separations,	$(2^2)(1)^2$	$(2^2)(1^2)$ $(21^2)(2)$	$(2^2 1)(1)$	$(2^2 1^2)$
	Assemblage,	$(2^2)(1)^2$	$-2(2^2)(1^2)$ $-2(21^2)(2)$	$-(2^2 1)(1)$	$+6(2^2 1^2)$

and the identity of the two assemblages, viz.

$$(31)(1^2) - 4(31^2)(1) + 9(31^3) = (2^2)(1)^2 - 2(2^2)(1^2) - 2(21^2)(2) - (2^2 1)(1) + 6(2^2 1^2)$$

is easily established; each is, in fact, equal to

$$(42) - 2(41^2) - 3(321).$$

172. The theorem is not restricted to positive integers, but there seems to be no advantage to be gained practically in extending it to the most general case.

§12.

173. In the foregoing section another law of symmetry was incidentally met with. As it is of importance, I propose to further examine it.

From the relation

$$C_{p_1}^{\pi_1} C_{p_2}^{\pi_2} \ldots\ldots = \ldots\ldots + L B_{s_1}^{\sigma_1} B_{s_2}^{\sigma_2} \ldots\ldots + \ldots\ldots, \qquad \text{(I)}$$

is derived the operator relation

$${}_\beta G_{p_1}^{\pi_1}\, {}_\beta G_{p_2}^{\pi_2} \ldots\ldots = \ldots\ldots + L\, {}_\gamma G_{s_1}^{\sigma_1}\, {}_\gamma G_{s_2}^{\sigma_2} \ldots\ldots + \ldots\ldots,$$

and operating with these two sides upon opposite sides of an *assumed* relation

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots\ldots)_\gamma = \ldots\ldots + A\,(p_1^{\pi_1} p_2^{\pi_2} \ldots\ldots)_\beta + \ldots\ldots \qquad \text{(II)}$$

we obtain at once

$$L = A,$$

so that a law of symmetry is involved in the two results

$$\begin{cases} C_{p_1}^{\pi_1} C_{p_2}^{\pi_2} \ldots\ldots = \ldots\ldots + L B_{s_1}^{\sigma_1} B_{s_2}^{\sigma_2} \ldots\ldots + \ldots\ldots, \\ (s_1^{\sigma_1} s_2^{\sigma_2} \ldots\ldots)_\gamma = \ldots\ldots + L\,(p_1^{\pi_1} p_2^{\pi_2} \ldots\ldots)_\beta + \ldots\ldots \end{cases}$$

By direct multiplication of the product

$$C_{p_1}^{\pi_1} C_{p_2}^{\pi_2} \ldots\ldots,$$

we observe that L is composed of separations of the symmetric function

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots\ldots)_\alpha,$$

of specification $(p_1^{\pi_1} p_2^{\pi_2} \ldots\ldots)$;

L is in fact composed of the tabular assemblages of separations which appear in the tables of the first two memoirs.

174. Hence in the result

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots\ldots)_\gamma = \ldots\ldots + L\,(p_1^{\pi_1} p_2^{\pi_2} \ldots\ldots)_\beta + \ldots\ldots$$

L is composed of tabular separations of the symmetric function

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots\ldots)_\alpha$$

of specification $(p_1^{\pi_1} p_2^{\pi_2} \ldots\ldots)$.

175. It is now easy to form any function

$$(s_1^{\sigma_1}s_2^{\sigma_2}\ldots\ldots)_\gamma.$$

Example I.

To form $(1^30)_\gamma$, first write down all the separations of (1^30), and underneath them their respective specifications.

Separations:	$(1)^3(0)$,	$(10)(1)^2$,	$(1^2)(1)(0)$,	$(1^2)(10)$,	$(1^3)(0)$,	$(1^20)(1)$,	(1^30),
Specifications:	(1^30)	(1^3)	(210)	(21)	(30)	(21)	(3).

We have now to form the C products corresponding to the several specifications and therein pick out the terms involving the B-product $B_1^3B_0$.

Thus
$$\begin{aligned}
C_1^3C_0 &= \ldots\ldots + (1)^3(0)\,B_1^3B_0 &&+\ldots\ldots,\\
C_1^3 &= \ldots\ldots + 3\,(1)^2(10)\,B_1^3B_0 &&+\ldots\ldots,\\
C_2C_1C_0 &= \ldots\ldots + (1^2)(1)(0)\,B_1^3B_0 &&+\ldots\ldots,\\
C_2C_1 &= \ldots\ldots + \{(1^2)(10) + (1^20)(1)\}\,B_1^3B_0 &&+\ldots\ldots,\\
C_3C_0 &= \ldots\ldots + (1^3)(0)\,B_1^3B_0 &&+\ldots\ldots,\\
C_3 &= \ldots\ldots + (1^30)\,B_1^3B_0 &&+\ldots\ldots
\end{aligned}$$

Hence

$$\begin{aligned}
(1^30)_\gamma = (1)^3(0)(1^30)_\beta + 3\,(1)^2(10)(1^3)_\beta + (1^2)(1)(0)(210)_\beta&\\
+ \{(1^2)(10) + (1^20)(1)\}\,(21)_\beta + (1^3)(0)(30)_\beta + (1^30)(3)_\beta&.
\end{aligned}$$

Observe that from this point of view $(1^30)_\gamma$ is a generating function for the tabular separations of symmetric function (1^30).

176. If we now take the operator relation

$$\begin{aligned}
1 + {}_\beta G_0 + {}_\beta G_1 y \quad + \ldots\ldots = \prod_{s=1}^{s=n}\Big(1 + {}_\gamma G_0 + \alpha_s\,{}_\gamma G_1 y \quad &+ \ldots\ldots\\
+ {}_\beta G_{-1}\frac{1}{y} + \ldots\ldots \qquad\qquad + \frac{1}{\alpha_s}\,{}_\gamma G_{-1}\frac{1}{y} &+ \ldots\ldots\Big)
\end{aligned}$$

we obtain
$$\begin{aligned}
{}_\beta G_0 &= (0)\,{}_\gamma G_0 &&+\ldots\ldots,\\
{}_\beta G_1 &= (1)\,{}_\gamma G_1 + (10)\,{}_\gamma G_1\,{}_\gamma G_0 &&+\ldots\ldots,
\end{aligned}$$

where only those terms which are significant for present purposes have been retained.

Operating with these relations on opposite sides of the expression for $(1^30)_\gamma$, we have

$$(0)(1^3)_\gamma = (1)^3(0)(1^3)_\beta + (1^2)(1)(0)(21)_\beta + (1^3)(0)(3)_\beta,$$

which is
$$(1^3)_\gamma = (1)^3(1^3)_\beta + (1^2)(1)(21)_\beta + (1^3)(3)_\beta$$

and

$$(1)(1^2 0)_\gamma + (10)(1^2)_\gamma = (1)\{(1)^2(0)(1^2 0)_\beta + 2\,(1)(10)(1^2)_\beta + (1^2)(0)(20)_\beta + (1^2 0)(2)_\beta\}$$
$$+ (10)\{(1)^2(1^2)_\beta + (1^2)(2)_\beta\},$$

which is true, because the process explained gives

$$(1^2 0)_\gamma = (1)^2(0)(1^2 0)_\beta + 2\,(1)(10)(1^2)_\beta + (1^2)(0)(20)_\beta + (1^2 0)(2)_\beta$$

and

$$(1^2)_\gamma = (1)^2(1^2)_\beta + (1^2)(2)_\beta .$$

This example will serve to show the method of utilizing the operators for the derivation of new results from those already obtained.

177. The most important cases are those which have reference merely to partitions composed of parts which are positive, non-zero, integers.

We may write down a table of weight 4.

$$\begin{aligned}
(4)_\gamma &= (4)(4)_\beta ,\\
(31)_\gamma &= (31)(4)_\beta + (3)(1)(31)_\beta ,\\
(2^2)_\gamma &= (2^2)(4)_\beta + (2)^2(2^2)_\beta ,\\
(21^2)_\gamma &= (21^2)(4)_\beta + (21)(1)(31)_\beta + 2\,(2)(1^2)(2^2)_\beta + (2)(1)^2(21^2)_\beta ,\\
(1^4)_\gamma &= (1^4)(4)_\beta + (1^3)(1)(31)_\beta + (1^2)^2(2^2)_\beta + (1^2)(1)^2(21^2)_\beta + (1)^4(1^4)_\beta ,
\end{aligned}$$

and thence we may immediately write down the corresponding operator relations, viz. writing in general

$$\frac{\overline{g_\lambda^l g_\mu^m \ldots\ldots}}{l!\ m!\ldots\ldots} = \overline{g_\lambda^{(l)} g_\mu^{(m)} \ldots\ldots} ,$$

we have

$$\begin{aligned}
{}_\beta g_4 &= (4)\,{}_\gamma g_4 ,\\
\overline{{}_\beta g_3\,{}_\beta g_1} &= (31)\,{}_\gamma g_4 + (3)(1)\,\overline{{}_\gamma g_3\,{}_\gamma g_1} ,\\
\overline{{}_\beta g_2^{(2)}} &= (2^2)\,{}_\gamma g_4 \qquad\qquad\qquad\qquad + (2)^2\,\overline{{}_\gamma g_2^{(2)}} ,\\
\overline{{}_\beta g_2\,{}_\beta g_1^{(2)}} &= (21^2)\,{}_\gamma g_4 + (21)(1)\,\overline{{}_\gamma g_3\,{}_\gamma g_1} + 2\,(2)(1^2)\,\overline{{}_\gamma g_2^{(2)}} + (2)(1)^2\,\overline{{}_\gamma g_2\,{}_\gamma g_1^{(2)}} ,\\
\overline{{}_\beta g_1^{(4)}} &= (1^4)\,{}_\gamma g_4 + (1^3)(1)\,\overline{{}_\gamma g_3\,{}_\gamma g_1} + (1^2)^2\,\overline{{}_\gamma g_2^{(2)}} + (1^2)(1)^2\,\overline{{}_\gamma g_2\,{}_\gamma g_1^{(2)}} + (1)^4\,{}_\gamma g_1^{(4)} ,
\end{aligned}$$

and so in general it is clear that on the right-hand side we must obtain each tabular separation of the partition of the β operation on the left-hand side attached to a symbolic product of γ operations of the corresponding specification. These results may be utilized in a variety of ways, though at present I do not stop to further discuss them.

§13.

The Multiplication of Symmetric Functions.

178. Professor Cayley has, in this Journal (Vol. VII, p. 2), laid down an algorithm for the multiplication of two symmetric functions which can be sym-

bolically expressed by partitions composed of positive integers. Moreover, I have, in the Messenger of Mathematics (Vol. XIV, p. 164), shown how to apply differential operators to the same purpose.

The method of operators may be used when the product contains any number whatever of functions, the coefficient of each term in the result being obtained by a separate and direct process.

Professor Cayley's process on the other hand, while more simple in the case of a product of two functions, is not adapted when the number of functions exceeds two; it may be employed when the partitions contain, as well, zero and negative integer parts.

Of this I give two examples.

179. *Example* I. To multiply

$$\Sigma\alpha^3\beta\gamma^0\delta^{-1} \text{ and } \Sigma\alpha^{-1}\beta^{-1};$$

that is, to form the product

$$(310\bar{1})(\bar{1}^2),$$

3	1	0	$\bar{1}$	$\bar{1}\bar{1}$	F	$\div 2$	M	C
$\bar{1}$	$\bar{1}$				2	$(20^2\bar{1})$	2	2
$\bar{1}$		$\bar{1}$			2	$(21\bar{1}^2)$	2	2
$\bar{1}$			$\bar{1}$		2	$(210\bar{2})$	1	1
	$\bar{1}$	$\bar{1}$			2	$(30\bar{1}^2)$	2	2
	$\bar{1}$		$\bar{1}$		2	$(30^2\bar{2})$	2	2
		$\bar{1}$	$\bar{1}$		2	$(31\bar{1}\bar{2})$	1	1
$\bar{1}$				$\bar{1}$	2	$(210\bar{1}^2)$	2	2
	$\bar{1}$			$\bar{1}$	2	$(30^2\bar{1}^2)$	4	4
		$\bar{1}$		$\bar{1}$	2	$(31\bar{1}^3)$	6	6
			$\bar{1}$	$\bar{1}$	2	$(310\bar{1}\bar{2})$	1	1
				$\bar{1}\bar{1}$	1	$(310\bar{1}^3)$	6	3

where F means "frequency," M, multiplicity, and the coefficients in the column headed C are to be attached to the partitions in the same rows.

Reference should be made to Cayley (loc. cit.)

180. *Example* II. To multiply

$$\Sigma\alpha^0\beta^0\gamma^0 \text{ and } \Sigma\alpha^0\beta^0,$$

that is, to form the product

$$(0^3)(0^2),$$

we have

000	00	F	$\div 12$	M	C
00		6	(0^3)	6	3
0	0	6	(0^4)	24	12
	00	1	(0^5)	120	10

and the result is

$$(0^3)(0^2) = 3\,(0^3) + 12\,(0^4) + 10\,(0^5).$$

181. To establish the method of operators, suppose ϕ to be any rational integral function of $A_0, A_1, A_{-1}, A_2, A_{-2}, \ldots,$

and let

$$\phi = \phi_1\phi_2\phi_3 \ldots \phi_s;$$

if now we make the transformation of Art. 129, we shall have

$$\begin{aligned}
&\begin{pmatrix} 1 + G_0 + G_1\mu & + G_2\mu^2 & + \ldots \\ \quad + G_{-1}\dfrac{1}{\mu} & + G_{-2}\mu^2 & + \ldots \end{pmatrix}\phi \\
= &\begin{pmatrix} 1 + G_0 + G_1\mu & + G_2\mu^2 & + \ldots \\ \quad + G_{-1}\dfrac{1}{\mu} & + G_{-2}\dfrac{1}{\mu^2} & + \ldots \end{pmatrix}\phi_1 \\
\times &\begin{pmatrix} 1 + G_0 + G_1\mu & + G_2\mu^2 & + \ldots \\ \quad + G_{-1}\dfrac{1}{\mu} & + G_{-2}\dfrac{1}{\mu^2} & + \ldots \end{pmatrix}\phi_2 \\
\times &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\times &\begin{pmatrix} 1 + G_0 + G_1\mu & + G_2\mu^2 & + \ldots \\ \quad + G_{-1}\dfrac{1}{\mu} & + G_{-2}\dfrac{1}{\mu^2} & + \ldots \end{pmatrix}\phi_s
\end{aligned}$$

or

$$\begin{pmatrix} 1 + G_0 + G_1\mu & + G_2\mu^2 & + \dots \\ \quad + G_{-1}\frac{1}{\mu} & + G_{-2}\frac{1}{\mu^2} & + \dots \end{pmatrix}\phi$$

$$= \prod_{s=1}^{s=s}\begin{pmatrix} \phi_s + G_0\phi_s + G_1\phi_s\mu & + G_2\phi_s\mu^2 & + \dots \\ \quad + G_{-1}\phi_s\frac{1}{\mu} & + G_{-2}\phi_s\frac{1}{\mu^2} & + \dots \end{pmatrix}.$$

Whence, equating cofactors of like powers of μ, we obtain

$$G_\lambda\phi = \Sigma\Sigma G_{\lambda_1}\phi_{l_1}\,.\,G_{\lambda_2}\phi_{l_2}\,.\,G_{\lambda_3}\phi_{l_3}\,\dots\,,$$

where

(1). $(\lambda_1\lambda_2\lambda_3\dots)$ is any partition whatever of λ into positive, zero and negative integers.

(2). $\phi_{l_1}, \phi_{l_2}, \phi_{l_3}, \dots$ are different members of the set $\phi_1, \phi_2, \phi_3, \dots \phi_s$.

(3). The summation is taken for every partition of λ into positive, zero and negative parts.

(4). The summation is further taken for all the expressions obtained by permuting the numbers $l_1, l_2, l_3, \dots$ in all possible ways.

182. Hence if ϕ be a symmetric function expressed by a product of partitions, the operation G_λ is performed by abstracting every partition of λ in *all possible ways* from the product, one part at most being taken from each partition.

The process is in general simple enough and it must be made perfectly clear by a series of examples.

183. *Example* I. Let

$$\phi = (0^4)(0^3)(0^2).$$

Then

$$\begin{aligned} G_0\phi = \; & G_0(0^4)\,.\,(0^3)(0^2) + (0^4)\,.\,G_0(0^3)\,.\,(0^2) + (0^4)(0^3)\,.\,G_0(0^2) \\ & + G_0(0^4)\,.\,G_0(0^3)\,.\,(0^2) + G_0(0^4)\,.\,(0^3)\,.\,G_0(0^2) + (0^4)\,.\,G_0(0^3)\,.\,G_0(0^2) \\ & + G_0(0^4)\,.\,G_0(0^3)\,.\,G_0(0^2). \\ = \; & (0^3)(0^3)(0^2) + (0^4)(0^2)(0^2) + (0^4)(0^3)(0) \\ & + (0^3)(0^2)(0^2) + (0^3)(0^3)(0) + (0^4)(0^2)(0) \\ & + (0^3)(0^2)(0); \end{aligned}$$

where observe, in the first, second and third lines respectively, the partitions (0), (0^2), (0^3), each of which is a partition of zero, have been abstracted in all possible ways from the product in such wise that one part only is taken from each partition of the product.

184. *Example* II. Let

$$\phi = (0)^m;$$

then

$$G_0\phi = m\,(0)^{m-1} + \frac{1}{2!}\,m\,(m-1)(0)^{m-2} + \ldots . + \tfrac{1}{2}\,m\,(m-1)(0^2) + m\,(0) + 1,$$

the successive terms in the value of $G_0\phi$ corresponding to the abstraction of $(0), (0^2), \ldots . (0^{m-2}), (0^{m-1}), (0^m)$.

Thus $$G_0\,(0)^m = \{1 + (0)\}^m - (0)^m;$$

that is, $$G_0 n^m = (n+1)^m - n^m,$$

which is clearly right.

In general
$$\begin{aligned} G_0 f(n) &= f(n+1) - f(n) \\ &= (E-1)\,f(n) \end{aligned}$$

in the notation of the calculus of finite differences. Hence G_0 is precisely equivalent to the symbol Δ of the same calculus, or

$$G_0 f(n) = \Delta f(n).$$

185. *Example* III. Let

$$\phi = (2)(1)(3\bar{1})(0).$$

To operate with G_1 we have to consider the following partitions of unity, viz.

$$(2\bar{1}),\ (2\bar{1}0),\ (1),\ (10).$$

Hence $$G_1\phi = (1)(3)(0) + (1)(3) + (2)(3\bar{1})(0) + (2)(3\bar{1}).$$

186. *Example* IV. Coming now to multiplication, suppose we are required to find the coefficient of the term

$$(30^2\bar{2})$$

in the product $$(310\bar{1})(\bar{1}^2).$$

If $$(310\bar{1})(\bar{1}^2) = \ldots . + A\,(30^2\bar{2}) + \ldots . ,$$

then, operating throughout with $G_3 G_0^2 G_{-2}$, we find

$$G_3 G_0^2 G_{-2}\,(310\bar{1})(\bar{1}^2) = \ldots . + A + \ldots . ,$$

where on the right-hand side the only term which is a simple number is the sought coefficient A; we have then to operate upon the product with the operation $G_3 G_0^2 G_{-2}$ and the numerical term that issues will be the result we seek.

To carry this out, we find

$$G_{-2}(310\bar{1})(\bar{1}^2) = (310)(\bar{1}),$$
$$G_0 G_{-2}(310\bar{1})(\bar{1}^2) = (31)(\bar{1}) + (30),$$
$$G_0^2 G_{-2}(310\bar{1})(\bar{1}^2) = (3) + (3) = 2(3),$$
$$G_3 G_0^2 G_{-2}(310\bar{1})(\bar{1}^2) = 2.$$

Hence $$(310\bar{1})(\bar{1}^2) = \ldots. + 2(30^2\bar{2}) + \ldots.,$$

which agrees of course with the result obtained by Cayley's algorithm.

§14.

The Linear Partial Differential Operators of the Theory of Separations.

187. I propose to adapt the operations

$$g_0,\ g_1,\ g_{-1},\ g_2,\ g_{-2},\ \ldots.,$$

so that they may be performed on the expression of any symmetric function in terms of separations of a given partition.

It will be remembered that a partition is separated into partitions termed "separates." Any combination whatever of the parts of a partition may present itself as a separate. If the separable partition be

$$(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots.),$$

precisely $(l_1+1)(l_2+1)(l_3+1)\ldots. - 1$ distinct separates *may* occur.

It is necessary to consider all these separates as independent variables.

Let then $$(\ldots.\ 2^{p_2+\pi_2}1^{p_1+\pi_1}0^{p_0+\pi_0}\bar{1}^{p_{-1}+\pi_{-1}}\bar{2}^{p_{-2}+\pi_{-2}}\ldots.)$$

be any separate of a given separable partition

$$P,$$

then, by a known theorem,

$$g_s \equiv \Sigma g_s(\ldots 2^{p_2+\pi_2}1^{p_1+\pi_1}0^{p_0+\pi_0}\bar{1}^{p_{-1}+\pi_{-1}}\bar{2}^{p_{-2}+\pi_{-2}}\ldots)\,\partial_{(\ldots 2^{p_2+\pi_2}1^{p_1+\pi_1}0^{p_0+\pi_0}\bar{1}^{p_{-1}+\pi_{-1}}\bar{2}^{p_{-2}+\pi_{-2}}\ldots)},$$

the summation being in regard to every separate. Moreover, we have proved the relation

$$\frac{(-)^s}{s}g_s = \sum\frac{(-)^{\Sigma\pi}(\Sigma\pi-1)!}{\ldots.\ \pi_2!\ \pi_1!\ \pi_0!\ \pi_{-1}!\ \pi_{-2}!\ \ldots.}\ldots.\ G_2^{\pi_2}G_1^{\pi_1}G_0^{\pi_0}G_{-1}^{\pi_{-1}}G_{-2}^{\pi_{-2}}\ldots.,$$

when the summation is in regard to the solutions in positive, zero and negative integers of the indeterminate equation

$$\Sigma t\pi_t = s,$$

and also

$$\ldots\ldots G_2^{\pi_2} G_1^{\pi_1} G_0^{\pi_0} G_{-1}^{\pi_{-1}} G_{-2}^{\pi_{-2}} \ldots\ldots (\ldots\ldots 2^{p_2+\pi_2} 1^{p_1+\pi_1} 0^{p_0+\pi_0} \bar{1}^{p_{-1}+\pi_{-1}} \bar{2}^{p_{-2}+\pi_{-2}} \ldots\ldots)$$
$$= (\ldots\ldots 2^{p_2} 1^{p_1} 0^{p_0} \bar{1}^{p_{-1}} \bar{2}^{p_{-2}} \ldots\ldots).$$

188. Hence
$$\frac{(-)^s}{s} g_s$$
$$\equiv \sum\sum \frac{(-)^{\Sigma\pi}(\Sigma\pi - 1)!}{\ldots\ldots \pi_2!\, \pi_1!\, \pi_0!\, \pi_{-1}!\, \pi_{-2}! \ldots\ldots}$$
$$(\ldots\ldots 2^{p_2} 1^{p_1} 0^{p_0} \bar{1}^{p_{-1}} \bar{2}^{p_{-2}} \ldots\ldots)\, \partial_{(\ldots\ldots 2^{p_2+\pi_2} 1^{p_1+\pi_1} 0^{p_0+\pi_0} \bar{1}^{p_{-1}+\pi_{-1}} \bar{2}^{p_{-2}+\pi_{-2}} \ldots\ldots)},$$

the summation being in regard

(1) to every separate;

(2) to every solution of the indeterminate equation $\Sigma t\pi_t = s$ in positive, zero, and negative integers.

189. To take a concrete case, consider the separable partition
$$(21^{n-2});$$
we then have
$$g_1 \equiv \partial_{(1)} + (1)\,\partial_{(1^2)} + (1^2)\,\partial_{(1^3)} + \ldots\ldots + (1^{n-3})\,\partial_{(1^{n-2})}$$
$$+ (2)\,\partial_{(21)} + (21)\,\partial_{(21^2)} + (21^2)\,\partial_{(21^3)} + \ldots\ldots + (21^{n-3})\,\partial_{(21^{n-2})},$$
$$g_s \equiv \partial_{(1^s)} + (1)\,\partial_{(1^{s+1})} + (1^2)\,\partial_{(1^{s+2})} + \ldots\ldots + (1^{n-s-2})\,\partial_{(1^{n-2})}$$
$$+ (2)\,\partial_{(21^s)} + (21)\,\partial_{(21^{s+1})} + \ldots\ldots + (21^{n-s-2})\,\partial_{(21^{n-2})}$$
$$- s\{\partial_{(21^{s-2})} + (1)\,\partial_{(21^{s-1})} + \ldots\ldots + (1^{n-s})\,\partial_{(21^{n-2})}\},$$
relations which are obtained at once from the general formula.

190. Again, consider the separable partition
$$(321^2);$$
the separates are

$$\begin{array}{llll} (1) & , (2) & , (3) & , \\ (1^2) & , (21) & , (31) & , (32), \\ (21^2) & , (31^2), & (321), & \\ (321^2); & & & \end{array}$$

then
$$g_1 \equiv \partial_{(1)} + (1)\,\partial_{(1^2)} + (2)\,\partial_{(21)} + (21)\,\partial_{(21^2)} + (3)\,\partial_{(31)} + (31)\,\partial_{(31^2)} + (32)\,\partial_{(321)} + (321)\,\partial_{(321^2)};$$
wherein only those separates which contain the part unity occur as independent variables. Also
$$g_2 \equiv \partial_{(1^2)} + (2)\,\partial_{(21^2)} + (3)\,\partial_{(31^2)} + (32)\,\partial_{(321^2)}$$
$$- 2\{\partial_{(2)} + (1)\,\partial_{(21)} + (1^2)\,\partial_{(21^2)} + (3)\,\partial_{(32)} + (31)\,\partial_{(321)} + (31^2)\,\partial_{(321^2)}\},$$

wherein the operation is comprised of two portions corresponding to the partitions (1^2) and (2) of the number 2.

In the first and second portions respectively the independent variables contain the partitions (1^2) and (2).

This should be compared with the known formula

$$s_2 = a_1^2 - 2a_2.$$

Similarly

$$\begin{aligned} g_3 \equiv & -3\{\partial_{(21)} + (1)\,\partial_{(21^2)} + (3)\,\partial_{(321)} + (31)\,\partial_{(321^2)}\} \\ & + 3\{\partial_{(3)} + (1)\,\partial_{(31)} + (1^2)\,\partial_{(31^2)} + (2)\,\partial_{(32)} + (21)\,\partial_{(321)} + (21^2)\,\partial_{(321^2)}\} \end{aligned}$$

in correspondence with

$$s_3 = \ldots\ldots - 3a_2a_1 + 3a_3,$$

and

$$\begin{aligned} g_4 \equiv & -4\partial_{(21^2)} \\ & + 4\{\partial_{(31)} + (1)\,\partial_{(31^2)} + (2)\,\partial_{(321)} + (21)\,\partial_{(321^2)}\} \end{aligned}$$

in correspondence with

$$s_4 = -4a_2a_1^2 + 4a_3a_1 + \ldots\ldots$$

Also

$$\begin{aligned} g_5 &\equiv 5\{\partial_{(31^2)} + (2)\,\partial_{(321^2)}\} - 5\{\partial_{(32)} + (1)\,\partial_{(321)} + (1^2)\,\partial_{(321^2)}\}, \\ g_6 &\equiv -12\{\partial_{(321)} + (1)\,\partial_{(321^2)}\}, \\ g_7 &\equiv -21\partial_{(321^2)}, \end{aligned}$$

in correspondence respectively with

$$\begin{aligned} s_5 &= 5a_3a_1^2 + \ldots\ldots, \\ s_6 &= -12a_3a_2a_1 + \ldots\ldots, \\ s_7 &= -21a_3a_2a_1^2 + \ldots\ldots \end{aligned}$$

Any of these results may be easily verified. Thus from the first memoir

$$(7) = \ldots\ldots - \tfrac{1}{3}(321^2) + \ldots\ldots,$$

and since

$$g_7(7) = +7,$$

it is obvious that g_7 and $-21\partial_{(321^2)}$ are equivalent operations.

191. The formation of these operators is seen to be particularly simple in the case of a separable partition composed merely of positive (non-zero) integers. They are written down at once from the expression of Vandermonde for the sums of the powers of the quantities in terms of the elementary (that is, unitary) symmetric functions.

Let us now consider a separable partition which contains as well zero and negative integers.

For example, the partition

$$(210^2\bar{1})$$

of weight 2.

We must now consider the expressions of the sums of the powers in terms of the arguments

$$A_0,\ A_1,\ A_{-1},\ A_2,\ A_{-2},\ \ldots,$$

merely retaining necessary terms of the infinite expressions

$$\begin{aligned}
s_{-1} &= A_{-1} - A_0A_{-1} + A_0^2A_{-1} + \ldots,\\
s_0 &= A_0 - \tfrac{1}{2}A_0^2 - A_1A_{-1} + 2A_1A_0A_{-1} - 3A_1A_0^2A_{-1} + \ldots,\\
s_1 &= A_1 - A_1A_0 + A_1A_0^2 - A_2A_{-1} + 2A_2A_0A_{-1} - 3A_2A_0^2A_{-1} + \ldots,\\
s_2 &= -2A_2 + 2A_2A_0 - 2A_2A_0^2 - 4A_2A_1A_{-1} + 12A_2A_1A_0A_{-1} - 24A_2A_1A_0^2A_{-1} + \ldots,\\
s_3 &= -3A_2A_1 + 6A_2A_1A_0 - 9A_2A_1A_0^2 + \ldots.
\end{aligned}$$

192. Hence the following expressions for

$$g_{-1},\ g_0,\ g_1,\ g_2,\ g_3,$$

viz.

$$\begin{aligned}
g_{-1} \equiv \{&\partial_{(\bar{1})} + (0)\partial_{(0\bar{1})} + (1)\partial_{(1\bar{1})} + (2)\partial_{(2\bar{1})} + (0^2)\partial_{(0^2\bar{1})} + (10)\partial_{(10\bar{1})} + (20)\partial_{(20\bar{1})}\\
&+ (21)\partial_{(21\bar{1})} + (210)\partial_{(210\bar{1})} + (20^2)\partial_{(20^2\bar{1})} + (10^2)\partial_{(10^2\bar{1})} + (210^2)\partial_{(210^2\bar{1})}\}\\
-\{&\partial_{(0\bar{1})} + (0)\partial_{(0^2\bar{1})} + (1)\partial_{(10\bar{1})} + (2)\partial_{(20\bar{1})} + (10)\partial_{(10^2\bar{1})} + (20)\partial_{(20^2\bar{1})}\\
&+ (21)\partial_{(210\bar{1})} + (210)\partial_{(210^2\bar{1})}\}\\
+\{&\partial_{(0^2\bar{1})} + (1)\partial_{(10^2\bar{1})} + (2)\partial_{(20^2\bar{1})} + (21)\partial_{(210^2\bar{1})}\};\\
g_0 \equiv\ &\partial_{(0)} + (2)\partial_{(20)} + (1)\partial_{(10)} + (0)\partial_{(0^2)} + (\bar{1})\partial_{(0\bar{1})} + (21)\partial_{(210)}\\
&+ (0\bar{1})\partial_{(0^2\bar{1})} + (10)\partial_{(10^2)} + (20)\partial_{(20^2)} + (1\bar{1})\partial_{(10\bar{1})} + (2\bar{1})\partial_{(20\bar{1})}\\
&+ (210)\partial_{(210^2)} + (21\bar{1})\partial_{(210\bar{1})} + (20\bar{1})\partial_{(20^2\bar{1})} + (10\bar{1})\partial_{(10^2\bar{1})} + (210\bar{1})\partial_{(210^2\bar{1})}\\
-\tfrac{1}{2}\{&\partial_{(0^2)} + (1)\partial_{(10^2)} + (2)\partial_{(20^2)} + (\bar{1})\partial_{(0^2\bar{1})} + (21)\partial_{(210^2)}\\
&+ (2\bar{1})\partial_{(20^2\bar{1})} + (1\bar{1})\partial_{(10^2\bar{1})} + (21\bar{1})\partial_{(210^2\bar{1})}\}\\
-\{&\partial_{(1\bar{1})} + (0)\partial_{(10\bar{1})} + (0^2)\partial_{(10^2\bar{1})} + (2)\partial_{(21\bar{1})} + (20)\partial_{(210\bar{1})} + (20^2)\partial_{(210^2\bar{1})}\}\\
+2\{&\partial_{(10\bar{1})} + (0)\partial_{(10^2\bar{1})} + (2)\partial_{(210\bar{1})} + (20)\partial_{(210^2\bar{1})}\}\\
-3\{&\partial_{(10^2\bar{1})} + (2)\partial_{(210^2\bar{1})}\};\\
g_1 \equiv\ &\partial_{(1)} + (0)\partial_{(10)} + (\bar{1})\partial_{(1\bar{1})} + (2)\partial_{(21)} + (0^2)\partial_{(0^2 1)} + (0\bar{1})\partial_{(10\bar{1})} + (2\bar{1})\partial_{(21\bar{1})}\\
&+ (20)\partial_{(210)} + (0^2\bar{1})\partial_{(10^2\bar{1})} + (20\bar{1})\partial_{(210\bar{1})} + (20^2)\partial_{(210^2)} + (20^2\bar{1})\partial_{(210^2\bar{1})}\\
-\{&\partial_{(10)} + (\bar{1})\partial_{(10\bar{1})} + (0)\partial_{(10^2)} + (2)\partial_{(210)} + (0\bar{1})\partial_{(10^2\bar{1})} + (2\bar{1})\partial_{(210\bar{1})}\\
&+ (20)\partial_{(210^2)} + (20\bar{1})\partial_{(210^2\bar{1})}\}\\
+\{&\partial_{(10^2)} + (\bar{1})\partial_{(10^2\bar{1})} + (2)\partial_{(210^2)} + (2\bar{1})\partial_{(210^2\bar{1})}\}\\
-\{&\partial_{(2\bar{1})} + (0)\partial_{(20\bar{1})} + (1)\partial_{(21\bar{1})} + (0^2)\partial_{(20^2\bar{1})} + (10)\partial_{(210\bar{1})} + (10^2)\partial_{(210^2\bar{1})}\}\\
+2\{&\partial_{(20\bar{1})} + (0)\partial_{(20^2\bar{1})} + (1)\partial_{(210\bar{1})} + (10)\partial_{(210^2\bar{1})}\}\\
-3\{&\partial_{(20^2\bar{1})} + (1)\partial_{(210^2\bar{1})}\};
\end{aligned}$$

$$
\begin{aligned}
g_2 \equiv & -2\{\partial_{(2)} + (\bar{1})\,\partial_{(2\bar{1})} + (0)\,\partial_{(20)} + (1)\,\partial_{(21)} + (0\bar{1})\,\partial_{(20\bar{1})} + (0^2)\,\partial_{(20^2)} \\
& \qquad + (1\bar{1})\,\partial_{(21\bar{1})} + (10)\,\partial_{(210)} + (0^2\bar{1})\,\partial_{(20^2\bar{1})} + (10\bar{1})\,\partial_{(210\bar{1})} \\
& \qquad + (10^2)\,\partial_{(210^2)} + (10^2\bar{1})\,\partial_{(210^2\bar{1})}\} \\
& + 2\{\partial_{(20)} + (\bar{1})\,\partial_{(20\bar{1})} + (0)\,\partial_{(20^2)} + (1)\,\partial_{(210)} + (0\bar{1})\,\partial_{(20^2\bar{1})} \\
& \qquad + (1\bar{1})\,\partial_{(210\bar{1})} + (10)\,\partial_{(210^2)} + (10\bar{1})\,\partial_{(210^2\bar{1})}\} \\
& - 2\{\partial_{(20^2)} + (\bar{1})\,\partial_{(20^2\bar{1})} + (1)\,\partial_{(210^2)} + (1\bar{1})\,\partial_{(210^2\bar{1})}\} \\
& - 4\{\partial_{(21\bar{1})} + (0)\,\partial_{(210\bar{1})} + (0^2)\,\partial_{(210^2\bar{1})}\} \\
& + 12\{\partial_{(210\bar{1})} + (0)\,\partial_{(210^2\bar{1})}\} \\
& - 24\partial_{(210^2\bar{1})}, \\
g_3 \equiv & -3\{\partial_{(21)} + (\bar{1})\,\partial_{(21\bar{1})} + (0)\,\partial_{(210)} + (0\bar{1})\,\partial_{(210\bar{1})} + (0^2)\,\partial_{(210^2)} + (0^2\bar{1})\,\partial_{(210^2\bar{1})}\} \\
& + 6\{\partial_{(210)} + (\bar{1})\,\partial_{(210\bar{1})} + (0)\,\partial_{(210^2)} + (0\bar{1})\,\partial_{(210^2\bar{1})}\} \\
& - 9\{\partial_{(210^2)} + (\bar{1})\,\partial_{(210^2\bar{1})}\}.
\end{aligned}
$$

193. These operations appear to be somewhat complicated, but it will be seen subsequently that for practical use they may in general be broken up into effective fragments. Their application to calculations in the separation theory must be reserved for a future occasion; I hope then also to bring forward a new theorem of distribution of an extended character and to apply the method of this memoir to its analytical solution.

The main result was communicated by me verbally to the London Mathematical Society at its February meeting.

WOOLWICH, ENGLAND, *April* 24, 1889.

Remarque au sujet du théorème d'Euclide sur l'infinité du nombre des nombres premiers.

PAR M. JOSEPH PEROTT à *Gra-Thumiac* (*Morbihan*).

(*Suite.*)

11.

Soit u un nombre non supérieur à θ, nous désignerons le groupe formé par l'ensemble des solutions de l'égalité

$$x^{e^u} = 1 \qquad (\text{gr. } \Xi_\theta)$$

sous le nom de *sous-groupe principal de rang u* de Ξ_θ. Il est clair que ce sous-groupe est identique à Ξ_u. Les groupes

$$H_1, \; H_2, \ldots \ldots H_\theta$$

porteront le nom de *sous-groupes caractéristiques* de Ξ_θ. Quant aux nombres

$$m_1, \; m_2, \; \ldots \ldots m_\theta$$

nous les désignerons sous le nom de *nombres caractéristiques* de Ξ_θ.

Nous avons vu qu'il est toujours possible de trouver un système de m_1 éléments

$$a_1, \; a_2, \; \ldots \ldots a_{m_1}$$

tels que l'expression

$$a_1^{x_1} a_2^{x_2} \ldots \ldots a_{m_1}^{x_{m_1}} \qquad (\text{où } x_k = 0, 1, 2 \ldots \ldots v_k - 1)$$

soit susceptible de représenter tous les éléments du groupe Ξ_θ et chacun une seule fois. Un tel système d'éléments

$$a_1, \; a_2, \; \ldots \ldots a_{m_1}$$

portera le nom de *système de bases* du groupe Ξ_θ et chaque élément tel que a_k sera dit une des bases du système. Le nombre des bases d'un tel système est d'ailleurs constant et égal à m_1. En vertu de cette propriété nous dirons que le

groupe Ξ_θ est *m_1-base* ou *à m_1 bases.* Si $m_1 > 1$, on dira que le groupe Ξ_θ est *polybase* par opposition au cas où $m_1 = 1$ et dans quel cas le groupe Ξ_θ est dit *monobase.* On voit donc que tout groupe Ξ_θ qui n'est pas monobase est polybase. Nous avons montré encore que, d'une manière générale, si l'expression telle que

$$b_1^{y_1},\ b_2^{y_2},\ \ldots .\ b_n^{y_n},$$

où $b_1, b_2, \ldots . b_n$ désignent des éléments du groupe Ξ_θ, est susceptible de représenter tous les éléments du groupe Ξ_θ, on aura

$$n \geqq m_1.$$

Ranger les sous-groupes monobases d'ordre s du groupe Ξ_θ en classes suivant leurs portées respectives en les faisant émaner du groupe-unité, puis en faire émaner les sous-groupes monobases d'ordre s^2 rangés suivant leurs portées respectives, etc. (comme nous l'avons fait au §8), sera dit *ranger* (to array, aufstellen) le dit groupe Ξ_θ.

Soit A un sous-groupe de Ξ_θ et u son rang par rapport au nombre premier s, le groupe sera identique au groupe formé par l'ensemble des solutions de l'égalité

$$x^{s^u} = 1 \quad (\text{gr. } A)$$

et par suite tout à fait analogue au groupe Ξ_θ.

D'une manière générale, soit Φ un groupe eulérien, t un nombre premier et w un nombre entier quelconques, le groupe B formé par l'ensemble des solutions de l'égalité

$$x^{t^w} = 1 \quad (\text{gr. } \Phi)$$

sera tout à fait analogue au groupe Ξ_θ. En effet, soit ψ le rang du groupe Φ par rapport au nombre premier t. Si $\psi = 0$, le groupe B est identique au groupe-unité. Si $w \geqq \psi$ le groupe B est identique à l'ensemble des solutions de l'égalité

$$x^{t^\psi} = 1 \quad (\text{gr. } \Phi).$$

Enfin, si $w < \psi$ le groupe B est identique à l'ensemble des solutions de l'égalité

$$x^{t^w} = 1 \quad (\text{gr. } B).$$

Donc dans tous les cas le groupe B est analogue au groupe Ξ_θ, si l'on convient de considérer le groupe-unité comme appartenant à la catégorie des groupes Ξ_θ. On aura alors toujours

$$\Xi_0 = 1 \quad (\text{gr. } \Omega).$$

12.

Passons maintenant à la résolution de l'égalité

$$AX = B \qquad (\text{gr. } \Xi_\theta)$$

où A et B désignent deux sous-groupes donnés de Ξ_θ et X un sous-groupe cherché. Nous commencerons d'ailleurs par le cas particulier où

$$B = \Xi_\theta,$$

c. à-d. nous nous occuperons de la résolution de l'égalité

$$AX = \Xi_\theta \qquad (\text{gr. } \Xi_\theta).$$

Si l'on décompose A et X en un produit de groupes simples, l'égalité précédente donnera une décomposition de Ξ_θ en un produit de groupes simples. La décomposition de A ne doit donc contenir aucun groupe simple qui ne puisse figurer dans une décomposition de Ξ_θ. Cela revient à dire que tout groupe simple M d'ordre s, qui est sous-groupe tant de A que de Ξ_θ doit être de la même portée dans ces deux groupes.

Je me propose de faire voir que la condition précedente qui est nécessaire pour que l'égalité

$$AX = \Xi_\theta \qquad (\text{gr. } \Xi_\theta)$$

soit résoluble, est aussi suffisante.

Soit u le rang du groupe A par rapport au nombre premier s,

$$\Phi_1, \; \Phi_2, \; \ldots \; \Phi_u$$

ses groupes caractéristiques et

$$n_1, \; n_2, \; \ldots \; n_u$$

ses nombres caractéristiques. Il est clair que Φ_k, où $k \leqq u$, est un sous-groupe de H_k; je dis que tout élément de Φ_k qui ne fait pas partie de Φ_{k+1}* ne peut faire partie de H_{k+1} non plus.

En effet, si un élément b faisait partie de Φ_k et de H_{k+1} sans faire partie de Φ_{k+1}, le groupe B formé par les éléments

$$1, \; b, \; b^2, \; \ldots \; b^{s-1}$$

* Pour $k = u$, le groupe Φ_{k+1} est supposé identique au groupe-unité; et de même $H_{\theta+1}$ est supposé identique au groupe-unité.

serait de portée s^k dans le groupe A et d'une portée non inférieure à s^{k+1} dans le groupe Ξ_θ, contrairement à la supposition. L'inverse est aussi vraie, c. à-d. si tout élément de Φ_k (pour toute valeur de k depuis 1 jusqu'à u) qui ne fait pas partie de Φ_{k+1} ne fait pas partie de H_{k+1} non plus, tout sous-groupe simple d'ordre s de A et de Ξ_θ est de la même portée dans les deux groupes. En effet, toute base b d'un tel sous-groupe simple B fera partie de groupes tels que Φ_l et H_l sans faire partie de Φ_{l+1} ni de H_{l+1}; le groupe B sera donc de la portée s^l dans les deux groupes.

Désignons par A_1 le sous-groupe principal de rang 1 du groupe A, on aura

$$A_1 = \Phi_1.$$

Cela étant ainsi, il y aura trois cas à considérer

1). $u < \theta$;
2). $u = \theta$, mais Φ_θ différent de H_θ;
3). $\Phi_\theta = H_\theta$.

Dans le premier cas, soit B^{s^θ} un groupe monobase d'ordre s^θ et B^s son émettant d'ordre s, je dis que le groupe B^s n'aura de commun avec le groupe A_1 que l'élément-unité. En effet, si les groupes A_1 et B^s avaient en commun un élément b différent de l'élément-unité, cet élement b ferait nécessairement partie de H_θ et de Φ_k par exemple où $k \leqq u$, sans faire partie de Φ_{k+1}. Le groupe B^s serait donc de la portée s^k où $k < \theta$ dans A et de la portée s^θ dans Ξ_θ contrairement à la supposition.

Je pose donc $$A' = AB^{s^\theta},$$

le groupe A' sera de rang θ. Soient

$$\Phi'_1,\ \Phi'_2,\ \ldots\ \Phi'_\theta$$

ses groupes caractéristiques, on aura

$$\Phi'_k = B^s$$

pour $u < k \leqq \theta$ et $$\Phi'_k = \Phi_k B^s$$

pour $k \leqq u$.

Je dis que tout élément c qui fait partie d'un groupe caractéristique de A' tel que Φ'_k sans faire partie de Φ'_{k+1}, fera bien partie de H_k, mais non de H_{k+1}. Pour $k = \theta$, on a $$H_{k+1} = \Phi'_{k+1} = 1$$

et pour $\theta > k > u$ $$\Phi'_k = \Phi'_{k+1}.$$

de sorte qu'il suffit de considérer le cas où $k \leq u$. L'élément c peut s'obtenir par la composition d'un élément b du groupe B^s avec un élément d du groupe Φ_k ne faisant pas partie de Φ_{k+1} et par suite de H_{k+1} non plus. Comme l'élément b fait partie du groupe H_{k+1}, il est clair que l'élément c n'en fera pas partie.

2^{me} cas. Dans ce cas soit b un élément du groupe H_θ ne faisant pas partie de Φ_θ, le groupe monobase B^s auquel b sert de base n'aura de commun avec le groupe Φ_θ que l'élément-unité. Soit B^{s^θ} un émanant d'ordre s^θ du groupe B^s; je dis d'abord que B^s n'aura de commun avec le groupe A_1 que l'élément-unité. En effet, si B^s et A_1 avaient en commun un élément différent de l'élément-unité, le groupe B^s serait un sous-groupe de A_1. Le groupe B^s ferait donc partie d'un groupe tel que Φ_k où $k < \theta$ sans faire partie du groupe Φ_{k+1}; il serait donc de la portée s^k dans le groupe A et de la portée s^θ dans le groupe Ξ_θ contrairement à la supposition. Cela étant ainsi, posons

$$A' = AB^{s^\theta}$$

et soient

$$\Phi'_1, \Phi'_2, \ldots . \Phi'_\theta$$

les groupes caractéristiques de A'. On aura

$$\Phi'_k = \Phi_k B^s$$

pour toutes les valeurs de k depuis 1 jusqu'à θ. Je dis maintenant que tout élément c qui fait partie de Φ'_k sans faire partie de Φ'_{k+1} ne peut faire partie de H_{k+1} non plus. En effet, l'élément c peut s'obtenir par la composition d'un élément b du groupe B^s avec un élément d du groupe Φ_k ne faisant pas partie de Φ_{k+1}. Or l'élément b fait partie du groupe H_{k+1} tandis que l'élément d n'en fait pas partie, l'élément $c = bd$ n'en fait donc pas partie non plus. Le groupe A' est donc analogue au groupe A en ce que tout sous-groupe monobase d'ordre s des groupes A' et Ξ_θ est de la même portée dans les deux groupes.

3^{me} cas. On a

$$\Phi_k = H_k$$

pour toutes les valeurs de k depuis θ jusqu'à l, où $l < \theta$. Si $l = 1$, les groupes A et Ξ_θ sont de même ordre et par suite identiques. Supposons donc $l > 1$. Soit b un élément du groupe H_{l-1} ne faisant pas partie de Φ_{l-1}, les éléments

$$1, b, b^2, \ldots . b^{s-1}$$

formeront un groupe B^s qui fera partie de H_{l-1}, mais pas de Φ_{l-1}. Je dis que les groupes A_1 et B^s n'ont de commun que l'élément-unité. En effet, si les groupes

A_1 et B^s avaient en commun un élément c appartenant à l'exposant s, tout le groupe B^s ferait partie d'un groupe tel que Φ_m où $m < l-1$ sans faire partie de Φ_{m+1} et le groupe B^s serait de la portée s^m dans le groupe A et de la portée s^{l-1} dans le groupe Ξ_θ, contrairement à la supposition. Soit $B^{s^{l-1}}$ un émanant d'ordre s^{l-1} du groupe B^s, je pose

$$A' = AB^{s^{l-1}}$$

et je désigne par

$$\Phi'_1, \Phi'_2, \ldots . \Phi'_\theta$$

les groupes caractéristiques de A'. On aura évidemment

$$\Phi'_k = \Phi_k$$

pour $\theta \geqq k \geqq l$ et

$$\Phi'_k = \Phi_k B^s$$

pour $k < l$. Il est clair que tout élément d'un groupe tel que Φ'_{k-1} qui ne fait pas partie de Φ'_k, ne peut faire partie de H_k non plus tant que $k > l$. Si $k = l$ soit c un élément du groupe Φ'_{k-1} ne faisant pas partie de Φ'_k. Cet élément s'obtient par la composition d'un élément b du groupe B^s avec un élément d du groupe Φ_{k-1}. Si $b = 1$, c est un élément du groupe Φ_{k-1} ne faisant pas partie de Φ_k et par suite de H_k non plus. Si b est différent de 1, l'élément c ne fait pas partie de Φ_{k-1} et par suite de $\Phi_k = H_k$ non plus. Soit maintenant $k < l$, l'élément c s'obtient alors par la composition d'un élément b du groupe B^s avec un élément d du groupe Φ_{k-1} ne faisant pas partie de Φ_k et de H_k non plus. L'élément b fait partie du groupe H_k tandis que l'élément d n'en fait pas partie; il en résulte que l'élément $c = bd$ n'en fait pas partie. On voit que toutes les fois que A est un groupe différent de Ξ_θ et tel que tout sous-groupe monobase d'ordre s de A et de Ξ_θ est de la même portée dans les deux groupes, on peut multiplier le dit groupe A par un groupe monobase de manière à obtenir un groupe A' jouissant encore de la même propriété que tout sous-groupe monobase d'ordre s des groupes A' et Ξ_θ est de la même portée dans ces deux groupes. Comme cela ne peut aller à l'infini, il est clair qu'après une ou plusieurs transformations, on finira par obtenir le groupe Ξ_θ. Le produit de tous les groupes monobases par lesquels on aura multiplié A et les groupes transformés, donnera une solution de l'égalité

$$AX = \Xi_\theta \quad (\text{gr. } \Xi_\theta).$$

Un groupe A pour lequel une telle égalité est possible portera le nom de *diviseur* de Ξ_θ. Le produit de A et X donnant Ξ_θ, chacun des groupes A et X portera le nom de *facteur complémentaire* de l'autre (par rapport au produit Ξ_θ).

Passons maintenant à la détermination du nombre des solutions de l'égalité

$$AX = \Xi_\theta \quad (\text{gr. } \Xi_\theta).$$

Soit v le rang d'une solution X de l'égalité

$$AX = \Xi_0 \quad (\text{gr. } \Xi_0)$$

et

$$r_1, r_2, \ldots . r_v$$

ses nombres caractéristiques. Si u est inférieur à θ, je poserai

$$n_k = 0,$$

pour $u < k \leqq \theta$ et de même si $v < \theta$, je poserai

$$r_k = 0$$

pour $v < k \leqq \theta$. Cela étant ainsi, on aura évidemment

$$\begin{gathered} n_1 + r_1 = m_1, \\ n_2 + r_2 = m_2, \\ \cdots\cdots\cdots \\ n_k + r_k = m_k, \\ \cdots\cdots\cdots \\ n_\theta + r_\theta = m_\theta. \end{gathered}$$

On voit que les nombres

$$r_1, r_2, \ldots . r_\theta$$

sont indépendants de la solution X qu'on a choisie pour les déterminer et même qu'il est facile de les déterminer sans connaître aucune solution de l'égalité

$$AX = \Xi_\theta \quad (\text{gr. } \Xi_\theta).$$

Je pose en outre $\quad m_{\theta+1} = n_{\theta+1} = r_{\theta+1} = 0;$

les différences

$$\begin{gathered} m_k - m_{k+1}, \\ n_k - n_{k+1}, \\ r_k - r_{k+1}, \end{gathered}$$

où $0 < k < \theta + 1$, ne seront jamais négatives. On aura d'ailleurs

$$m_k - m_{k+1} = (n_k - n_{k+1}) + (r_k - r_{k+1}).$$

Désignons par X_1 le sous-groupe principal de rang 1 d'une solution X, on aura évidemment

$$A_1 X_1 = \Xi_1 \quad (\text{gr. } \Xi_\theta).$$

Nous dirons qu'une telle solution X_1 de l'égalité

$$A_1 X_1 = \Xi_1 \quad (\text{gr. } \Xi_\theta)$$

correspond à la solution X de l'égalité

$$AX = \Xi_\theta \quad (\text{gr. } \Xi_\theta).$$

On voit qu'à toute solution X de l'égalité

$$AX = \Xi_\theta \quad (\text{gr. } \Xi_\theta)$$

correspond une solution bien déterminée de l'égalité

$$AX_1 = \Xi_1 \quad (\text{gr. } \Xi_\theta).$$

Nous dirons inversement que la solution X de l'égalité

$$AX = \Xi_\theta \quad (\text{gr. } \Xi_\theta)$$

correspond à la solution X_1 de l'égalité

$$A_1X_1 = \Xi_1 \quad (\text{gr. } \Xi_\theta).$$

La condition nécessaire et suffisante pour qu'il existe une solution X de l'égalité

$$AX = \Xi_\theta \quad (\text{gr. } \Xi_\theta)$$

correspondant à une solution donnée X_1 de l'égalité

$$AX_1 = \Xi_1 \quad (\text{gr. } \Xi_\theta)$$

consiste en ce que X_1 doit être décomposable en un produit de groupes monobases renfermant $r_k - r_{k+1}$ groupes monobases de portée s^k pour toute valeur de k depuis 1 jusqu'à θ. Une telle solution X_1 de l'égalité

$$A_1X_1 = \Xi_1 \quad (\text{gr. } \Xi_\theta)$$

portera le nom de solution *convenable* de l'égalité

$$A_1X_1 = \Xi_1 \quad (\text{gr. } \Xi_\theta).$$

Une décomposition (X_1) d'une solution convenable X_1 renfermant $s_k - s_{k+1}$ groupes de portée s^k sera nécessairement convenable par rapport à un certain groupe X auquel X_1 servira de sous-groupe principal de rang 1; nous désignerons une telle décomposition (X_1) d'une solution convenable X_1 de l'egalité

$$AX_1 = \Xi_1 \quad (\text{gr. } \Xi_\theta)$$

sous le nom de *décomposition convenable.*

Cherchons maintenant le nombre de toutes les décompositions convenables de toutes les solutions convenables de l'égalité

$$A_1X_1 = \Xi_1 \quad (\text{gr. } \Xi_\theta).$$

Soient (A_1') une décomposition convenable de A_1 par rapport au groupe A et (X_1) une décomposition convenable d'une solution convenable de l'égalité

$$A_1 X_1 = \Xi_1 \quad (\text{gr. } \Xi_\theta)$$

on aura évidemment

$$(A_1)(X_1) = (\Xi_1)$$

où (Ξ_1) est une décomposition convenable de Ξ_1 par rapport au groupe Ξ_θ. Nous voulons exprimer par l'égalité précédente qu'en prenant tous les facteurs de (A_1) et tous les facteurs de (X_1) on aura tous les facteurs de (Ξ_1). Si l'on remplace (A_1) par une autre décomposition $[A_1]$ de A_1 convenable par rapport au groupe A, on aura encore

$$[A_1](X_1) = [\Xi_1]$$

où $[\Xi_1]$ désigne une décomposition de Ξ_1 convenable par rapport au groupe Ξ_θ. La nouvelle égalité doit être d'ailleurs entendue dans le même sens que la précédente. Ce fait est indiqué par la suppression de la parenthèse (gr. Ξ_θ). On peut donc obtenir toutes les décompositions convenables de toutes les solutions convenables de l'égalité

$$A_1 X_1 = \Xi_1 \quad (\text{gr. } \Xi_\theta)$$

par la considération d'une égalité telle que

$$(A_1)(X_1) = (\Xi_1)$$

où (A_1) désigne une *certaine* décomposition convenable de A_1 par rapport au groupe A. Étant donnée une décomposition convenable (A_1) de A_1 (par rapport à A), pour parfaire une décomposition convenable de Ξ_1 (par rapport à Ξ_θ), il faut ajouter, en général, $r_k - r_{k+1}$ groupes monobases d'ordre s^k. Le produit explicite des groupes monobases qu'on aura ajoutés donnera une décomposition convenable d'une certaine solution convenable X_1 de l'égalité

$$A X_1 = \Xi_1 \quad (\text{gr. } \Xi_0)$$

et il est clair que toutes les décompositions convenables de toutes les solutions convenables de l'égalité

$$A X_1 = \Xi_1 \quad (\text{gr. } \Xi_\theta)$$

peuvent s'obtenir de cette manière. Il s'ensuit que le nombre de toutes les décompositions convenables de toutes les solutions convenables de l'égalité

$$A X_1 = \Xi_1 \quad (\text{gr. } \Xi_\theta)$$

est égal à

$$\prod_{k=1}^{k=\theta} \frac{\dfrac{s^{m_k} - s^{n_k + r_{k+1}}}{s-1} \cdot \dfrac{s^{m_k} - s^{n_k + r_{k+1} + 1}}{s-1} \cdots \dfrac{s^{m_k} - s^{m_k - 1}}{s-1}}{(r_k - r_{k+1})!}.$$

où, toutes les fois que r_k est égal à r_{k+1}, il faut remplacer le facteur correspondant

$$\frac{\dfrac{s^{m_k}-s^{n_k+r_{k+1}}}{s-1}\cdot\dfrac{s^{m_k}-s^{n_k+r_{k+1}+1}}{s-1}\cdots\dfrac{s^{m_k}-s^{m_k-1}}{s-1}}{(r_k-r_{k+1})!}$$

par l'unité. Or le nombre des décompositions convenables d'une solution convenable de

$$A_1X_1=\Xi_1 \quad (\text{gr. } \Xi_\theta)$$

est égal à

$$\prod_{k=1}^{k=\theta}\frac{\dfrac{s^{r_k}-1}{s-1}\cdot\dfrac{s^{r_k}-s}{s-1}\cdots\dfrac{s^{r_k}-s^{r_k-1}}{s-1}}{(r_k-r_{k+1})!}$$

où, toutes les fois que $r_k=r_{k+1}$ il faut remplacer le facteur correspondant

$$\frac{\dfrac{s^{r_k}-1}{s-1}\cdot\dfrac{s^{r_k}-s}{s-1}\cdots\dfrac{s^{r_k}-s^{r_k-1}}{s-1}}{(r_k-r_{k+1})!}$$

par l'unité. Le quotient des deux nombres que nous venons d'obtenir, donnera le nombre des solutions convenables de égalité

$$A_1X_1=\Xi_1 \quad (\text{gr. } \Xi_\theta).$$

Si dans une décomposition convenable (X_1) d'une solution convenable X_1 de l'égalité

$$A_1X_1=\Xi_1 \quad (\text{gr. } \Xi_\theta)$$

on remplace chaque facteur par un de ses émanants de l'ordre maximum, on obtiendra une décomposition d'une solution X de l'égalité

$$AX=\Xi_\theta \quad (\text{gr. } \Xi_\theta).$$

En prenant la décomposition (X_1) de X_1 comme point de départ, on peut obtenir toutes les décompositions correspondantes de X. Ces décompositions seront au nombre de

$$\prod_{k=2}^{k=v} s^{(r_{k-1}+r_{k-2}+\ldots\, r_1-k+1)(r_k-r_{k+1})}.$$

Or le nombre de toutes les décompositions des solutions de l'égalité

$$AX=\Xi_\theta \quad (\text{gr. } \Xi_\theta)$$

qu'on peut obtenir en prenant (X_1) comme point de départ est égal à

$$\prod_{k=2}^{k=\theta} s^{(m_{k-1}+m_{k-2}+\ldots\, m_1-k+1)(r_k-r_{k+1})}.$$

Le nombre des solutions de l'égalité

$$AX=\Xi_\theta \quad (\text{gr. } \Xi_\theta)$$

qui correspondent à une solution convenable de

$$A_1X_1 = \Xi_1 \quad (\text{gr. } \Xi_\theta)$$

est donc égal à

$$\frac{\prod\limits_{k=2}^{k=\theta} s^{(m_{k-1}+m_{k-2}+\ldots m_1-k+1)(r_k-r_{k+1})}}{\prod\limits_{k=2}^{k=v} s^{(r_{k-1}+r_{k-2}+\ldots r_1-k+1)(r_k-r_{k+1})}}.$$

Il est clair, en effet, qu'étant donnée une solution X_1 de l'égalité

$$A_1X_1 = \Xi_1 \quad (\text{gr. } \Xi_\theta)$$

la décomposition convenable (X_1) de X_1 qu'on prend pour point de départ est indifférente—on obtiendra toujours les mêmes solutions de

$$AX = \Xi_\theta \quad (\text{gr. } \Xi_\theta).$$

Pour obtenir le nombre des solutions de l'égalite

$$AX = \Xi_\theta \quad (\text{gr. } \Xi_\theta)$$

il suffit de multiplier le nombre des solutions convenables de l'égalité

$$A_1X_1 = \Xi_1 \quad (\text{gr. } \Xi_\theta)$$

par le nombre des solutions de l'égalité

$$AX = \Xi_\theta \quad (\text{gr. } \Xi_\theta)$$

qui correspondent à une solution convenable de l'égalité

$$A_1X_1 = \Xi_1 \quad (\text{gr. } \Xi_\theta).$$

L'égalité générale $\qquad AX = B \quad (\text{gr. } \Xi_\theta)$

pourra se traiter de la même manière, car si B est de rang w par rapport au nombre premier s, le groupe B est identique à l'ensemble des solutions de l'égalité $\qquad x^{s^w} = 1 \quad (\text{gr. } B)$

et par suite analogue en tout au groupe Ξ_θ.

13.

Abordons maintenant l'égalité bilinéaire

$$XY = \Xi_\theta \quad (\text{gr. } \Xi_\theta)$$

où X et Y sont deux sous-groupes cherchés. Soit v un nombre entier non supérieur à θ et

$$n_1, n_2, \ldots . n_v,$$

v nombres satisfaisant aux conditions

$$n_{k+1} \leqq n_k \quad (k = 1, 2, \ldots . v-1),$$
$$0 < n_v \leqq m_v$$
$$nk - n_{k+1} \leqq m_k - m_{k+1} \quad (k = 1, 2, \ldots . v-1).$$

Posons
$$n_k = 0 \quad (k = v+1, v+2, \ldots . \theta+1)$$
on aura
$$n_{k+1} \leqq n_k \quad (k = 1, 2, \ldots . \theta),$$
$$n_k - n_{k+1} \leqq m_k - m_{k+1} \quad (k = 1, 2, \ldots . \theta).$$

Les nombres r_k définis par les égalités

$$r_k = m_k - n_k \quad (k = 1, 2, \ldots . \theta+1)$$

satisferont alors aux conditions

$$r_{k+1} \leqq r_k \quad (k = 1, 2, \ldots . \theta)$$
$$r_k - r_{k+1} \leqq m_k - m_{k+1} \quad (k = 1, 2, \ldots . \theta).$$

Désignons maintenant par $\psi_\theta(m_k, n_k)$ le nombre des solutions de l'égalité

$$XY = \Xi_\theta \quad (\text{gr. } \Xi_\theta)$$

où X et Y admettent respectivement les n et les r comme nombres caractéristiques. Soit (Ξ_θ) une décomposition de Ξ_θ en groupes simples, je prends dans cette décomposition $n_k - n_{k+1}$ facteurs d'ordre s^k (pour $k = 1, 2, \ldots . \theta$) et je désigne par (X) le produit explicite de tous ces facteurs, tandis que (Y) désignera le produit explicite de tous les autres facteurs de (Ξ_θ); on aura alors

$$(X)(Y) = (\Xi_\theta)$$

ce qui veut dire que (Ξ_θ) renfermera tous les facteurs de (X) et de (Y) et n'en renfermera pas d'autres. Si l'on désigne par X le produit implicite de tous les facteurs de (X) et par Y le produit implicite de tous les facteurs de (Y), on aura bien

$$XY = \Xi_\theta \quad (\text{gr. } \Xi_\theta).$$

Pour chaque décomposition donnée (Ξ_θ) de Ξ_θ le nombre des égalités analogues à

$$(X)(Y) = (\Xi_\theta)$$

sera égal à
$$\prod_{k=1}^{k=\theta} \frac{(m_k - m_{k+1})!}{(n_k - n_{k+1})!\,(r_k - r_{k+1})!}.$$

Disons d'ailleurs une fois pour toutes que l'expression $0!$ aura toujours la valeur 1 dans toutes nos formules. Dans le cas où l'on a $n_k = r_k$ $(k = 1, 2, \ldots . \theta)$ à toute égalité telle que

$$(X)(Y) = (\Xi_\theta)$$

correspondra une égalité

$$(Y)(X) = (\Xi_\theta).$$

Désignons maintenant par $\tau_\theta(m_k)$ le nombre des décompositions de Ξ_θ en groupes simples et posons pour toute valeur de h supérieure à θ,

$$m_h = 0$$

et

$$\tau_h(m_k) = \tau_\theta(m_k);$$

nous aurons

$$\psi_\theta(m_k, n_k) = \frac{\tau_\theta(m_k)}{\tau_\theta(n_k) \times \tau_\theta(r_k)} \prod_{k=1}^{k=\theta} \frac{(m_k - m_{k+1})!}{(n_k - n_{k+1})!\,(r_k - r_{k+1})!}.$$

La valeur de $\tau_\theta(m_k)$ que nous avons obtenue au §9 devient, après quelques réductions

$$\tau_\theta(m_k) = s^{\frac{1}{2}(m_1^2 - m_1) + \sum\limits_{k=1}^{k=\theta} m_k m_{k+1} - \sum\limits_{k=2}^{k=\theta} m_k}$$
$$\times \prod_{k=1}^{k=\theta} \frac{\left(\frac{s^{m_k - m_{k+1}} - 1}{s - 1}\right)\left(\frac{s^{m_k - m_{k+1} - 1} - 1}{s - 1}\right) \ldots . \left(\frac{s - 1}{s - 1}\right)}{(m_k - m_{k+1})!}$$

d'où

$$\psi_\theta(m_k, n_k) = s^{n_1 r_1 + \sum\limits_{k=1}^{k=\theta} (n_k r_{k+1} + r_k n_{k+1})}$$
$$\times \prod_{k=1}^{k=\theta} \frac{(s^{m_k - m_{k+1}} - 1)(s^{m_k - m_{k+1} - 1} - 1) \ldots . (s^{r_k - r_{k+1} + 1} - 1)}{(s^{n_k - n_{k+1}} - 1)(s^{n_k - n_{k+1} - 1} - 1) \ldots . (s - 1)}.$$

On sait que Gauss a montré que chaque facteur du produit que nous venons d'écrire est une fonction entière de s. Cherchons maintenant le nombre des diviseurs de Ξ_θ admettant les nombres caractéristiques

$$n_1, n_2, \ldots . n_\theta.$$

Soit A un tel diviseur et A_1 son sous-groupe principal de rang 1, une décomposition (A_1) de A_1 convenable par rapport à A, contiendra $n_k - n_{k+1}$ groupes de portée s^k. Inversement, si l'on forme un sous-groupe B_1 de rang 1 susceptible d'une décomposition renfermant $n_k - n_{k+1}$ groupes simples de portée

$$s^k \quad (k = 1, 2, \ldots . \theta)$$

ayant pour produit un groupe premier* à H_{k+1} et que l'on remplace chaque facteur d'une telle décomposition par un de ses émanants de l'ordre le plus élevé; on obtiendra comme produit un diviseur B de Ξ_θ admettant les nombres caractéristiques

$$n_1, n_2, \ldots . n_\theta .$$

Cela étant ainsi, le nombre de toutes les décompositions convenables des sous-groupes principaux A_1 de rang 1 de tous les diviseurs A de Ξ_θ admettant les nombres caractéristiques

$$n_1, n_2, \ldots . n_\theta$$

sera égal à

$$\prod_{k=1}^{k=\theta} \frac{\left(\frac{s^{m_k}-s^{m_{k+1}}}{s-1}\right)\left(\frac{s^{m_k}-s^{m_{k+1}+1}}{s-1}\right)\cdots\left(\frac{s^{m_k}-s^{r_{k+1}+n_k-1}}{s-1}\right)}{(n_k-n_{k+1})!} .$$

Or le nombre des décompositions de A_1 convenables par rapport à A, est égal à

$$\prod_{k=1}^{k=\theta} \frac{\left(\frac{s^{n_k}-s^{n_{k+1}}}{s-1}\right)\left(\frac{s^{n_k}-s^{n_{k+1}+1}}{s-1}\right)\cdots\left(\frac{s^{n_k}-s^{n_k-1}}{s-1}\right)}{(n_k-n_{k+1})!} .$$

Le nombre des groupes A_1 sera donc égal à

$$\prod_{k=1}^{k=\theta} \left(\frac{s^{m_k}-s^{m_{k+1}}}{s^{n_k}-s^{n_{k+1}}}\right)\left(\frac{s^{m_k}-s^{m_{k+1}+1}}{s^{n_k}-s^{n_{k+1}+1}}\right)\cdots\left(\frac{s^{m_k}-s^{r_{k+1}+n_k-1}}{s^{n_k}-s^{n_k-1}}\right).$$

Quant au nombre des diviseurs A ayant un A_1 donné pour sous-groupe principal de rang 1, il est égal à

$$\frac{\prod_{k=2}^{k=\theta} s^{(m_{k-1}+m_{k-2}+\ldots . m_1-k+1)(n_k-n_{k+1})}}{\prod_{k=2}^{k=\theta} s^{(n_{k-1}+n_{k-2}+\ldots . n_1-k+1)(n_k-n_{k+1})}}$$

$$= \prod_{k=2}^{k=\theta} s^{(r_{k-1}+r_{k-2}+\ldots . r_1)(n_k-n_{k+1})}.$$

Si l'on désigne par $\phi_\theta(m_k, n_k)$ le nombre des diviseurs A de Ξ_θ admettant les nombres caractéristiques

$$n_1, n_2, \ldots . n_\theta ,$$

---

* C. à-d. que le dit produit et le groupe H_{k+1} ne doivent avoir en commun que l'élément-unité. En vertu de cette condition qui doit avoir lieu pour toute valeur de k depuis 1 jusqu'à θ, le groupe B est un diviseur de Ξ_θ. Inversement toute décomposition de B_1 convenable par rapport au diviseur B doit remplir cette condition.

on aura par conséquent

$$\phi_\theta(m_k,\, n_k) = \prod_{k=1}^{k=\theta}\left(\frac{s^{m_k-m_{k+1}}-1}{s^{n_k-n_{k+1}}-1}\right)\left(\frac{s^{m_k-m_{k+1}-1}-1}{s^{n_k-n_{k+1}-1}-1}\right)\cdots\cdots\left(\frac{s^{r_k-r_{k+1}+1}-1}{s-1}\right)$$

$$\times \prod_{k=1}^{k=\theta} s^{(n_k-n_{k+1})(r_{k+1}+r_{k-1}+r_{k-2}+\cdots\cdots r_2+r_1)}$$

$$= s^{\sum\limits_{k=1}^{k=\theta}(n_k r_{k+1}+r_k n_{k+1})-\sum\limits_{k=1}^{k=\theta} n_{k+1}r_{k+1}}$$

$$\times \prod_{k=1}^{k=\theta}\left(\frac{s^{m_k-m_{k+1}}-1}{s^{n_k-n_{k+1}}-1}\right)\left(\frac{s^{m_k-m_{k+1}-1}-1}{s^{n_k-n_{k+1}-1}-1}\right)\cdots\cdots\left(\frac{s^{r_k-r_{k+1}+1}-1}{s-1}\right).$$

De même, si l'on désigne par $\chi_\theta(m_k,\, n_k)$ le nombre des solutions de l'égalite

$$AX = \Xi_\theta \qquad (\text{gr. } \Xi_\theta)$$

où A est un diviseur donnée de Ξ_θ admettant les nombres caractéristiques

$$n_1,\, n_2,\, \ldots\ldots n_\theta,$$

on aura, après quelques réductions,

$$\chi_\theta(m_k,\, n_k) = \prod_{s=1}^{s=\theta} s^{(r_k-r_{k+1})(n_k+n_{k-1}+n_{k-2}+\cdots\cdots n_2+n_1)}$$

$$= s^{\sum\limits_{k=1}^{k=\theta} n\, r_k}.$$

Voici, en résumé, les valeurs de nos quatre fonctions arithmétiques

$$\tau_\theta(m_k) = s^{\frac{1}{2}(m_1^2-m_1)+\sum\limits_{k=1}^{k=\theta} m_k m_{k+1}-\sum\limits_{k=2}^{k=\theta} m_k}$$

$$\times \prod_{k=1}^{k=\theta}\frac{\left(\frac{s^{m_k-m_{k+1}}-1}{s-1}\right)\left(\frac{s^{m_k-m_{k+1}-1}-1}{s-1}\right)\cdots\cdots\left(\frac{s-1}{s-1}\right)}{(m_k-m_{k+1})!},$$

$$\phi_\theta(m_k,\, n_k) = s^{\sum\limits_{k=1}^{k=\theta}(n_k r_{k+1}+r_k n_{k+1})-\sum\limits_{k=1}^{k=\theta} n_{k+1}r_{k+1}}$$

$$\times \prod_{k=1}^{k=\theta}\left(\frac{s^{m_k-m_{k+1}}-1}{s^{n_k-n_{k+1}}-1}\right)\left(\frac{s^{m_k-m_{k+1}-1}-1}{s^{n_k-n_{k+1}-1}-1}\right)\cdots\cdots\left(\frac{s^{r_k-r_{k+1}+1}-1}{s-1}\right),$$

$$\chi_\theta(m_k,\, n_k) = s^{\sum\limits_{k=1}^{k=\theta} n_k r_k},$$

$$\psi_\theta(m_k,\, n_k) = s^{n_1 r_1+\sum\limits_{k=1}^{k=\theta}(n_k r_{k+1}+r_k n_{k+1})}$$

$$\times \prod_{k=1}^{k=\theta}\left(\frac{s^{m_k-m_{k+1}}-1}{s^{n_k-n_{k+1}}-1}\right)\left(\frac{s^{m_k-m_{k+1}-1}-1}{s^{n_k-n_{k+1}-1}-1}\right)\cdots\cdots\left(\frac{s^{r_k-r_{k+1}+1}-1}{s-1}\right).$$

Le nombre des solutions de l'égalité

$$X'Y = \Xi_\theta \quad (\text{gr. } \Xi_\theta)$$

où X' est un diviseur donné de Ξ_θ admettant les nombres caractéristiques

$$n_1, n_2, \ldots . n_\theta,$$

étant égal à $\chi_\theta(m_k, n_k)$, il est clair qu'on aura

$$\psi_\theta(m_k, n_k) = \phi_\theta(m_k, n_k)\,\chi_\theta(m_k, n_k).$$

Cette relation se réduit à une identité si l'on y substitue les valeurs de $\phi_\theta(m_k, n_k)$, $\chi_\theta(m_k, n_k)$ et $\psi_\theta(m_k, n_k)$. Le nombre de toutes les solutions de l'égalité

$$XY = \Xi_\theta \quad (\text{gr. } \Xi_\theta)$$

est exprimé par la somme

$$\sum_{(n_k)} \psi_\theta(m_k, n_k)$$

qu'il faut étendre à tous les systèmes de nombres caractéristiques

$$n_1, n_2, \ldots . n_\theta$$

vérifiant les conditions

$$(n_k - n_{k+1}) \leqq (m_k - m_{k+1}) \quad (k = 1, 2, \ldots . \theta).$$

L'égalité bilinéaire $\quad XY = A \quad (\text{gr. } \Xi_\theta)$

où A est un sous-groupe de Ξ_θ est analogue en tout à l'égalité bilinéaire

$$XY = \Xi_\theta \quad (\text{gr. } \Xi_\theta)$$

qui a fait l'objet de ce paragraphe.

14.

Le groupe Ω étant un groupe eulérien quelconque, nous avons défini le groupe Ξ_θ comme l'ensemble des solutions de l'égalité

$$x^{s^\theta} = 1 \quad (\text{gr. } \Omega)$$

où θ est le rang du groupe Ω par rapport à un nombre premier donné s. Tout groupe eulérien qui peut être obtenu d'une manière analogue sera désigné provisoirement sous le nom de groupe Ξ_θ. Un groupe Ξ_θ est dit *monostélèche* ou *polystélèche* suivant que le nombre de ses sous-groupes d'ordre s est égal ou supérieur à l'unité. Quand nous voudrons préciser d'avantage, nous dirons qu'un groupe Ξ_θ est $\frac{s^{m_1}-1}{s-1}$ *stélèche*. On voit que pour qu'un groupe Ξ_θ soit monostélèche il

faut et il suffit qu'on ait

$$m_1 = 1$$

de sorte que tout groupe Ξ_θ monostélèche est monobase et inversement. Un groupe Ξ_θ tel que tout sous-groupe monobase de ce groupe, sauf le groupe-unité, n'a qu'un seul et unique émanant de tout ordre non supérieur à sa portée, est dit *aclone*. Pour qu'un groupe Ξ_θ soit aclone il faut et il suffit qu'on ait

1) soit $\theta = 1$, et dans ce cas tout sous-groupe monobase de Ξ_θ, sauf le groupe-unité, n'aura qu'un seul et unique émanant;

2) soit, quand $\theta > 1$,

$$m_1 = 1$$

de sorte que le groupe sera monobase.

Les groupes Ξ_θ aclones peuvent donc être partagés en deux classes.

1) Les groupes aclones polystélèches. Ce seront tous les groupes Ξ_θ de rang 1, sauf le groupe d'ordre s.

2). Les groupes aclones monostélèches qui comprennent tous les groupes Ξ_θ monobases et inversement tout groupe Ξ_θ monobase est un groupe aclone monostélèche. Comme le nombre des solutions de l'égalité

$$x^s = 1 \quad (\text{gr. } \Xi_\theta)$$

est égal à s^{m_1}, il est clair que cette égalité n'admet que s solutions dans le cas où le groupe Ξ_θ est monobase et inversement si cette égalité n'admet que s solutions le groupe Ξ_θ est monobase. Les groupes de rang 1 ont été étudiés à part dans le §7.

Tout groupe Ξ_θ tel que tous ses sous-groupes monobases sont d'une même portée, est dit *isoclone*. Il est clair que la portée en question ne peut être que s^θ. Pour qu'un groupe Ξ_θ soit isoclone, il faut et il suffit qu'on ait

$$m_1 = m_2 = \ldots\ldots = m_\theta.$$

Comme cas particuliers de groupes Ξ_θ isoclones, on a pour $\theta = 1$ tous les groupes Ξ_θ de rang 1 et pour $m_1 = 1$ tous les groupes Ξ_θ monobases. On peut pousser jusqu'au bout la théorie des groupes isoclones avant d'aborder celle des groupes Ξ_θ quelconques. Nous verrons bientôt que tout groupe Ξ_θ est décomposable en un produit de groupes isoclones de rangs différents. Une telle décomposition correspond à la décomposition d'un nombre entier en un produit de puissances de nombre premier premières entre elles tandis que la décomposition en groupes simples correspond à la décomposition d'un nombre en un produit de nombres premiers. Il faut bien se garder de confondre ces deux décompositions bien distinctes.

15.

Si le groupe A contient tous les éléments du groupe B, on dit que le groupe A est un surgroupe du groupe B et que le groupe B est un sous-groupe du groupe A. Si A est un surgroupe de B et B est un surgroupe de C, le groupe A est un surgroupe de C et de même pour un nombre quelconque de groupes A, B, C, D, E, etc. Si le groupe A est un sous-groupe de B et B est un sous-groupe de C, le groupe A est un sous-groupe de C et de même pour un nombre quelconque de groupes. Si C est un sous-groupe tant de A que de B, on dit que C est un sous-groupe commun de A et B. L'ensemble de tous les éléments que A et B ont en commun forment un groupe D. En effet, deux éléments quelconques a et b de cet ensemble donnent par la composition l'élément ab qui fait partie tant de A que de B et par suite de D. Tout sous-groupe commun C de A et B est un sous-groupe de D qui peut pour cette raison recevoir le nom *du plus grand* commun sous-groupe de A et B*. Deux groupes A et B dont le plus grand commun sous-groupe est le groupe-unité sont dits *premiers entre eux*. On peut aussi dire que A est premier à B ou que B est premier à A. Inversement tout sous-groupe de D est un sous-groupe commun de A et B. De même les éléments que trois groupes quelconques A, B et C ont en commun forment un groupe D et tout groupe F qui est un sous-groupe de A, B et C en même temps est un sous-groupe de D. Un groupe tel que F porte le nom de sous-groupe commun de A, B et C. Le groupe D est le plus grand commun sous-groupe de A, B et C. Inversement tout sous-groupe de D est un sous-groupe commun de A, B et C. On étendra sans peine ces considérations à un nombre quelconque de groupes.

Si les groupes A et B sont tous les deux sous-groupes d'un groupe C, le groupe C est dit surgroupe commun de A et B. Le groupe C devra naturellement contenir tous les éléments qu'on peut obtenir en composant (gr. C) un élément du groupe A avec un élément du groupe B. L'ensemble de tous les éléments du groupe C qu'on peut obtenir en composant (gr. C) un élément de groupe A avec un élément du groupe B formeront un groupe S qui sera un sous-groupe de C et un surgroupe commun de A et B. Tout sous-groupe de C et surgroupe commun de A et B est un surgroupe de S. Le groupe S est donc dit *le plus petit commun surgroupe de A et B* (dans le groupe C). On définira de la

* Adjectif peut propre que nous gardons pour ne pas introduire trop de néologismes.

même manière un surgroupe commun R d'un nombre quelconque de groupes A, B, C, D, E, le plus petit commun surgroupe S des groupes A, B, C, D, E (dans le groupe R) et l'on fera voir que tout sous-groupe H de R qui est un surgroupe de A, B, C, D et E est un surgroupe de S. Inversement tout sous-groupe de R et surgroupe de S est un surgroupe A, B, C, D et E.

Si l'égalité
$$AX = B \quad (\text{gr. } \Xi_\theta)$$
est résoluble, on dit que A est un *diviseur* de B et que B est un *multiple* de A. Il est clair que tant A que X seront des sous-groupes de B.

Si les égalités
$$AX = B \quad (\text{gr. } \Xi_\theta),$$
$$AY = C \quad (\text{gr. } \Xi_\theta)$$
sont résolubles toutes les deux, on dit que A est un *diviseur commun* de B et C.

De même, si les égalités
$$BX = A \quad (\text{gr. } A)$$
$$CY = A \quad (\text{gr. } A)$$
sont résolubles toutes les deux, on dit que A est un *multiple commun* de B et C.

16.

Tout groupe Ξ_θ monobase étant simple, il est clair qu'il n'admettra comme diviseurs que lui-même et le groupe-unité.

Tout sous-groupe B d'un groupe A de rang 1 est un diviseur de A. En effet, si le groupe B est identique au groupe A ou au groupe-unité, on peut considérer B comme un diviseur de A. Si B n'est identique ni au groupe-unité ni au groupe A, soit c_1 un élément du groupe A ne faisant pas partie de B, les éléments
$$c_1,\ c_2,\ \ldots\ c_1^{s-1}$$
ne feront pas partie de B non plus et par suite le groupe C_1 formé par les éléments
$$1,\ c_1,\ c_1^2,\ \ldots\ c_1^{s-1}$$
sera premier à B et le groupe BC_1 sera un sous-groupe de A. Si BC_1 est identique à A, on aura
$$BC_1 = A \quad (\text{gr. } A)$$
si non soit c_2 un élément de A ne faisant pas partie de BC_1, l'élément c_2 servira de base à un groupe C_2 qui sera premier à BC_1 et le groupe BC_1C_2 sera encore un sous-groupe de A. Si le groupe BC_1C_2 est identique à A, on aura
$$BC_1C_2 = A \quad (\text{gr. } A)$$

si non soit c_3 un élément de A ne faisant pas partie de BC_1C_2, etc. En continuant de la même manière on parviendra à une égalité telle que

$$BC_1C_2\ldots\ldots C_n = A \quad \text{(gr. } A\text{)}$$

d'où en posant

$$C_1C_2\ldots\ldots C_n = C \quad \text{(gr. } A\text{)},$$
$$BC = A \quad \text{(gr. } A\text{)}$$

ce qui prouve bien que B est un diviseur de A. Le produit de deux diviseurs d'un groupe de rang 1 sera un sous-groupe et par conséquent un diviseur de ce groupe. De même, tout surgroupe de rang 1 d'un groupe de rang 1 est un multiple de ce dernier groupe.

Le plus petit commun surgroupe C de deux groupes A et B de rang 1 est nécessairement un groupe de rang 1, car le produit de deux éléments appartenant à l'exposant 1 ou s est un élément appartenant à l'exposant 1 ou s. Le groupe C sera donc un multiple commun de A et B. Soient s^t, s^u, s^v les ordres des groupes A, B, C et s^w l'ordre du plus grand commun sous-groupe D de A et B, je dis qu'on aura

$$t + u = v + w.$$

En effet, posons

$$DD_1 = A \quad \text{(gr. } A\text{)},$$
$$DD_2 = B \quad \text{(gr. } B\text{)};$$

toute multiple de rang 1 de A et B sera un multiple de DD_1 et D_2 et inversement, car tout surgroupe de rang 1 des groupes DD_1 et D_2 qui sont premiers entre eux, sera un surgroupe et par suite un multiple de DD_1D_2 et à plus forte raison de DD_2. On aura donc

$$DD_1D_2 = C \quad \text{(gr. } C\text{)}$$

et par conséquent

$$t + u = v + w.$$

Tout diviseur D d'un groupe isoclone A de rang θ est un groupe isoclone de rang θ et inversement tout sous-groupe isoclone de rang θ de A est un diviseur de ce dernier groupe. En effet, si le diviseur D n'était pas isoclone ou s'il était d'un rang inférieur à θ, il renfermerait un groupe monobase d'une portée inférieure à s^θ et par conséquent il ne pourrait être un diviseur de A. Pour démontrer la seconde partie de la proposition, désignons par A_1 et D_1 les sous-groupes principaux de rang 1 de A et D et posons

$$D_1B_1 = A_1 \quad \text{(gr. } A\text{)}.$$

On aura

$$DB = A \quad \text{(gr. } A\text{)},$$

où B est un sous-groupe isoclone de rang θ admettant B_1 comme sous-groupe

principal de rang 1. Le produit de deux diviseurs d'un groupe isoclone A de rang θ est un sous-groupe isoclone de rang θ et par suite un diviseur de A.

Le rang θ_3 du plus petit commun surgroupe de deux sous-groupes de Ξ_θ de rangs θ_1 et θ_2 respectivement peut être exprimé par la formule

$$\theta_3 = \frac{\theta_1\left[\frac{\theta_1}{\theta_2}\right] + \theta_2\left[\frac{\theta_2}{\theta_1}\right]}{\left[\frac{\theta_1}{\theta_2}\right] + \left[\frac{\theta_2}{\theta_1}\right]}$$

où $\left[\frac{\theta_1}{\theta_2}\right]$ est le plus grand entier contenu dans $\frac{\theta_1}{\theta_2}$ et $\left[\frac{\theta_2}{\theta_1}\right]$ le plus grand entier contenu dans $\frac{\theta_2}{\theta_1}$.

17.

Soient $\underline{A}$ et $\overline{A}$ deux sous-groupes quelconques de Ξ_θ et A leur plus petit commun surgroupe. Le groupe A sera nécessairement un sous-groupe de Ξ_θ. Désignons par

$$Z_1,\ Z_2,\ \ldots.\ Z_\theta$$

les sous-groupes caractéristiques et par

$$h_1,\ h_2,\ \ldots.\ h_\theta$$

les nombres caractéristiques de $\underline{A}$. Dans le cas où le groupe $\underline{A}$ est d'un rang u inférieur à θ, il faut poser

$$h_{u+1} = h_{u+2} = \ldots.\ h_\theta = 0.$$

Nous désignerons d'une manière analogue par

$$Z^1,\ Z^2,\ \ldots.\ Z^\theta$$

les groupes caractéristiques et par

$$h^1,\ h^2,\ \ldots.\ h^\theta$$

les nombres caractéristiques de $\overline{A}$. Enfin soient $\underline{\overline{A}}$ le plus grand commun sous-groupe de $\underline{A}$ et $\overline{A}$,

$$Z_1^1,\ Z_2^2,\ Z_3^3,\ \ldots.\ Z_\theta^\theta$$

ses sous-groupes caractéristiques et

$$h_1^1,\ h_2^2,\ h_3^3,\ \ldots.\ h_\theta^\theta$$

ses nombres caractéristiques. Soient, d'une maniére générale, $Z^k_{k,k+1}$ le plus grand commun sous-groupe de Z_k^k et Z_{k+1}, $Z_k^{k,k+1}$ le plus grand commun sous-groupe de Z_k^k et Z^{k+1}, enfin $Z_{k,k+1}^{k,k+1}$ le plus petit commun surgroupe de $Z^k_{k,k+1}$ et $Z_k^{k,k+1}$. Le groupe $Z_{k,k+1}^{k,k+1}$ sera un sous-groupe de Z_k^k. Désignons par

$$s^{h_k^{k,k+1}},\ s^{h^k_{k,k+1}},\ s^{h^{k,k+1}_{k,k+1}}$$

les ordres des groupes

$$Z_k^{k,\,k+1},\ Z^k_{k,\,k+1},\ Z^{k,\,k+1}_{k,\,k+1}.$$

Soient encore $Z^{k+1,\,k}_{k+1,\,k}$ le plus grand commun sous-groupe de $Z^k_{k,\,k+1}$ et $Z_k^{k,\,k+1}$ et $s^{k+1,\,k}_{k+1,\,k}$ son ordre, on aura la relation

$$h^{k+1,\,k}_{k+1,\,k} + h^{k,\,k+1}_{k,\,k+1} = h^k_{k,\,k+1} + h_k^{k,\,k+1}.$$

Je pose maintenant, pour abréger,

$$\begin{aligned} \Phi_{3k-2} &= Z^k_k, \\ g_{3k-2} &= h^k_k; \\ \Phi_{3k} &= Z^{k+1,\,k}_{k+1,\,k}, \\ g_{3k} &= h^{k+1,\,k}_{k+1,\,k}, \\ \Phi_{3k-1} &= Z^{k,\,k+1}_{k,\,k+1}, \\ g_{3k-1} &= h^{k,\,k+1}_{k,\,k+1}. \end{aligned}$$

Cela étant ainsi, un groupe tel que Φ_k sera un sous-groupe de tout groupe Φ_l où $l < k$ et surgroupe de tout groupe Φ_l où $l > k$. Il est à peine besoin de faire observer que des groupes voisins tels que Φ_k et Φ_{k+1} peuvent être identiques.

Soit D un diviseur commun quelconque de $\underline{A}$ et $\overline{A}$ et

$$\Psi_1,\ \Psi_2,\ \ldots\ \Psi_\theta$$

ses groupes caractéristiques. Désignons par D_k le plus grand commun sous-groupe de Φ_k et D et par s'_k son ordre, il est clair que D_k sera un sous-groupe de D_l quand $l < k$.

Le groupe Φ_1 est identique à l'ensemble des solutions de l'égalité

$$x^s = 1 \qquad (\text{gr. } \underline{A})$$

et par suite D_1 est identique au groupe formé par l'ensemble des solutions de l'égalité

$$x^s = 1 \qquad (\text{gr. } D)$$

on aura donc

$$D_1 = \Psi_1 \qquad (\text{gr. } \Xi_\theta).$$

Tout groupe Ψ_k est un sous-groupe de D_{3k-2}. En effet, Ψ_k est un sous-groupe de Z^k_k et D et par suite de D_{3k-2}. Je dis que de plus tout élément a qui fait partie* de Ψ_{k-1} sans faire partie de Ψ_k, fait partie de D_{3k-5} sans faire partie de D_{3k-2}. En effet, un tel élément a fait partie de Z_{k-1} et Z^{k-1} sans faire partie de Z_k ni de Z^k; il fera donc partie de Φ_{3k-5} sans faire partie de Φ_{3k-2} et par conséquent il fera partie de D_{3k-5} sans faire partie de D_{3k-2}. Cela étant ainsi, il est clair qu'on aura

$$\Psi_k = D_{3k-2} \qquad (\text{gr. } \Xi_\theta).$$

---

* D'où il résulte que D est un diviseur de $\underline{A}$.

En effet, si le groupe D_{3k-2} renfermait un élément b ne faisant pas partie de Ψ_k, cet élément ferait aussi partie de $D_1 = \Psi_1$ et par suite d'un groupe tel que Ψ_{l-1} sans faire partie de Ψ_l où $l \leq k$. L'élément b ferait donc partie de D_{3l-5} sans faire partie de D_{3l-2} et ne pourrait par conséquent faire partie de D_{3k-2}.

Décomposons maintenant le groupe $D_{3\theta-2}$ en groupes simples, puis achevons la décomposition de $D_{3\theta-3}$ en commençant par celle de $D_{3\theta-2}$, puis achevons la décomposition de $D_{3\theta-4}$ en commençant par celle de $D_{3\theta-3}$ et ainsi de suite jusqu'à ce que D_1 se trouve décomposé. Une telle décomposition de D_1 sera dite *convenable par rapport au diviseur D de $\underline{A}$ et $\overline{A}$.* Il est clair qu'elle sera aussi convenable par rapport à D dans le sens que nous avons donné à cette expression au §9. On aura d'ailleurs toujours

$$D_{3k-2} = D_{3k-3}, \qquad (\text{gr. } \Xi_\theta)$$

car si le groupe D_{3k-3} renfermait un élément b ne faisant pas partie de D_{3k-2}, cet élément servirait de base à un groupe B qui serait de rang s^{k-1} dans le groupe D et de rang s^k tant dans le groupe A que dans le groupe $\overline{A}$, ce qui est contraire à la supposition que D est un diviseur de $\underline{A}$ et $\overline{A}$.

Désignons par T_k le produit de tous les groupes simples qu'on a ajoutés à la décomposition de D_{k+1} afin d'achever celle de D_k et posons par extension

$$T_{3\theta-2} = D_{3\theta-2}.$$

Le groupe T_k portera le nom de *tronçon* (piece, Stück) n° k d'une décomposition du sous-groupe principal de rang 1 du diviseur D de $\underline{A}$ et $\overline{A}$. Tout tronçon dont le numéro est divisible par 3 sera d'ailleurs identique au groupe-unité. En posant

$$D_1 = T_1 T_2 \ldots . T_{3\theta-2}$$

on aura une décomposition de D_1 en tronçons. Je dis que tout tronçon n° $3k-2$ sera premier au groupe Φ^*_{3k-1}. En effet, tout élément a du tronçon T_{3k-2} autre que l'élément-unité, fait partie de D_{3k-2} sans faire partie de D_{3k-1} et par suite il fait partie de Φ_{3k-2} sans faire partie de Φ_{3k-1}.

Tout tronçon T_{3k-1} est premier tant au groupe $Z^k_{k,k+1}$ qu'au groupe $Z^{k,k+1}_k$. En effet, tout élément a du tronçon T_{3k-1} autre que l'élément-unité, fait partie de D_{3k-3} et D_{3k-2} sans faire partie de D_{3k} ni de D_{3k+1}; il fait donc partie de Z^k_k, Z_k et Z^k sans faire partie de Z_{k+1} ni de Z^{k+1}. Il ne peut donc faire partie ni de $Z^k_{k,k+1}$ ni de $Z^{k,k+1}_k$, ni à plus forte raison de $Z^{k+1,k}_{k+1,k} = \Phi_{3k}$. Quant au tronçon T_{3k} qui est identique au groupe-unité, on peut dire qu'il est premier

* Pour ne pas exclure le cas où $k = \theta$, on peut égaler $\Phi_{3\theta-1}$ au groupe-unité.

au groupe Φ_{3k+1}. Cela étant ainsi, nous désignerons sous le nom de tronçon de sous-groupe* de diviseur (de $\underline{A}$ et $\overline{A}$) n° $3k-2$ tout sous-groupe de Φ_{3k-2} qui est premier à Φ_{3k-1}. Je désignerai de même sous le nom de tronçon de sous-groupe de diviseur (de $\underline{A}$ et $\overline{A}$) n° $3k-1$ tout sous-groupe de Φ_{3k-1} qui est premier tant au groupe $Z^k_{k,\,k+1}$ qu'au groupe $Z_k^{k,\,k+1}$. Enfin je prendrai pour tronçon de sous-groupe de diviseur n° $3k$ le groupe-unité.

Si l'on prend un tronçon de sous-groupe de diviseur n° 1, puis un tronçon de sous-groupe de diviseur n° 2 et ainsi de suite, enfin un tronçon de sous-groupe de diviseur n° $3\theta-2$, on aura un *système* de tronçons de sous-groupe de diviseur. Nous avons vu que le sous-groupe principal de rang 1 de tout diviseur commun de $\underline{A}$ et $\overline{A}$ est décomposable en un produit de tronçons de sous-groupe de diviseur formant système. Je dis qu'inversement les $3\theta-2$ tronçons de tout système

$$\mathfrak{T}_1,\ \mathfrak{T}_2,\ \dots\ \mathfrak{T}_{3\theta-2}$$

peuvent être réunis en un produit $\mathfrak{D}_1$ qui servira de sous-groupe principal de rang 1 à un certain diviseur commun $\mathfrak{D}$ de $\underline{A}$ et $\overline{A}$. En effet, le tronçon $\mathfrak{T}_{3\theta-2}$ étant un sous-groupe de $\Phi_{3\theta-2}$ sera premier à $\mathfrak{T}_{3\theta-3}$, le produit $\mathfrak{T}_{3\theta-2}\mathfrak{T}_{3\theta-3}$ étant un sous-groupe de $\Phi_{3\theta-3}$ sera premier à $\mathfrak{T}_{3\theta-4}$, le produit $\mathfrak{T}_{3\theta-2}\mathfrak{T}_{3\theta-3}\mathfrak{T}_{3\theta-4}$ étant un sous-groupe de $\Phi_{3\theta-4}$ sera premier à $\mathfrak{T}_{3\theta-5}$ et ainsi de suite de sorte qu'on peut bien poser

$$\mathfrak{T}_1\mathfrak{T}_2 \dots \mathfrak{T}_{3\theta-2} = \mathfrak{D}_1 \quad (\text{gr. } \Xi_\theta).$$

En effet, décomposons chaque tronçon $\mathfrak{T}_k$ du système en facteurs simples

$$\mathfrak{T}_k = \mathfrak{T}_k^1\mathfrak{T}_k^2 \dots \mathfrak{T}_k^{e_k}$$

où s^{e_k} est l'ordre du tronçon $\mathfrak{T}_k$. Tout groupe simple $\mathfrak{T}_k^i$ d'une telle décomposition sera de la portée $s^{\left[\frac{k+2}{3}\right]}$ $\left(\text{où } \left[\frac{k+2}{3}\right] \text{ est le plus grand entier contenu dans } \frac{k+2}{3}\right)$ dans le groupe $\overline{A}$. Si l'on remplace chaque groupe simple de la décomposition de chaque tronçon du système par un de ses émanants de l'ordre maximum dans le groupe $\overline{A}$, on obtient comme produit un certain sous-groupe $\mathfrak{D}$ de $\overline{A}$ qui aura $\mathfrak{D}_1$ pour sous-groupe principal de rang 1. Soient

$$\Sigma_1,\ \Sigma_2,\ \dots\ \Sigma_\theta$$

les sous-groupes caractéristiques de $\mathfrak{D}$, on aura

$$\Sigma_k = \mathfrak{T}_{3k-2}\mathfrak{T}_{3k-1}\mathfrak{T}_{3k} \dots \mathfrak{T}_{3\theta-2} \quad (\text{gr. } \Xi_\theta).$$

*Il s'agit naturellement du sous-groupe principal de rang 1.

Tout groupe Σ_k sera un sous-groupe de $\Phi_{3k-2} = Z_k^k$ et par suite de Z_k et Z^k. Soit a un élément faisant partie de Σ_k sans faire partie de Σ_{k+1}, on peut faire

$$a = bcde \quad (\text{gr. } \underline{A})$$

où b est un élément de Σ_{k+1}, c un élément de $\mathfrak{T}_{3k}$ c. à-d. l'élément-unité, d un élément de $\mathfrak{T}_{3k-1}$ et e un élément de $\mathfrak{T}_{3k-2}$. L'élément cde et par suite l'élément de devra être différent de l'élément-unité. L'élément b fera toujours partie de Z_{k+1} et de Z^{k+1}. Dans le cas où $e = 1$, l'élément d fait partie de Z_k^k sans faire partie de $Z_{k,k+1}^k$ ni de $Z_k^{k,k+1}$, il ne pourra donc faire partie ni de Z_{k+1} ni de Z^{k+1}. L'élément $a = bd$ ne fera donc partie ni de Z_{k+1} ni de Z^{k+1}. Si e est différent de 1, l'élément bcd fait partie de $Z_{k,k+1}^{k,k+1}$ tandis que e n'en fait pas partie tout en faisant partie de Z_k^k; l'élément $a = bcde$ fait donc partie de Z_k^k sans faire partie de $Z_{k,k+1}^{k,k+1}$. A plus forte raison l'élément a ne fera partie ni de $Z_{k,k+1}^k$ ni de $Z_k^{k,k+1}$ et par suite ni de Z_{k+1} ni de Z^{k+1}. Le groupe $\mathfrak{D}$ est donc diviseur tant de $\underline{A}$ que de $\overline{A}$.

Nous dirons d'un tel diviseur $\mathfrak{D}$, qu'il correspond au dit système de tronçons

$$\mathfrak{T}_1, \mathfrak{T}_2, \ldots \mathfrak{T}_{3\theta-2}.$$

Plusieurs diviseurs peuvent correspondre au même système de tronçons.

Soit D un diviseur commun quelconque de $\underline{A}$ et $\overline{A}$, D_k le plus grand commun sous-groupe de D et Φ_k et s^{i_k} l'ordre de D_k, les groupes

$$D_1, D_2, D_3, \ldots D_{3\theta-2}$$

seront dits *groupes génériques* du diviseur D (de $\underline{A}$ et $\overline{A}$) et les nombres

$$i_1, i_2, \ldots i_{3\theta-2}$$

seront dits *nombres génériques* du diviseur D (de $\underline{A}$ et $\overline{A}$). Si l'on désigne par

$$d_1, d_2, \ldots d_\theta$$

les nombres caractéristiques de D, on aura la relation

$$d_k = i_{3k-2}.$$

Soit s^{e_k} l'ordre d'un tronçon n° k du sous-groupe principal de rang 1 du diviseur D, on aura

$$e_k = i_k - i_{k+1}.$$

Le nombre e_{3k} sera d'ailleurs toujours égal à zéro et l'ordre du diviseur D sera égal à

$$\prod_{k=1}^{k=3\theta-3} s^{e_k\left[\frac{k+2}{3}\right]}.$$

Cette expression montre que si l'on veut obtenir un diviseur D (de $\underline{A}$ et $\overline{A}$) de l'ordre maximum, disons un diviseur suprême (supreme, höchst) il faut porter au

maximum l'ordre de chaque tronçon. Désignons maintenant par s^{e_k} l'ordre de tout tronçon T_k de l'ordre maximum, et cherchons la valeur de e_k. Je dis d'abord que si l'on n'a pas

$$T_{3k-2}\Phi_{3k-1} = \Phi_{3k-2} \quad (\text{gr. } \Phi_{3k-2})$$

l'ordre du tronçon T_{3k-2} n'est pas maximum. En effet, si l'égalité précédente n'a pas lieu, soit b un élément du groupe Φ_{3k-2} ne faisant pas partie de $T_{3k-2}\Phi_{3k-1}$, les éléments

$$b,\ b^2,\ b^3,\ \ldots .\ b^{s-1}$$

ne feront pas partie de $T_{3k-2}\Phi_{3k-1}$. En effet, si un élément tel que b^t où $0 < t < s$ faisait partie du groupe $T_{3k-2}\Phi_{3k-1}$, il en serait de même de tout élément tel que b^{tt_1} et par suite aussi de l'élément b car on peut choisir t_1 de manière à avoir

$$b^{tt_1} = b \quad (\text{gr. } \Phi_{3k-2}).$$

Le groupe B formé par les éléments

$$1,\ b,\ b^2,\ \ldots .\ b^{s-1}$$

serait alors premier au groupe $T_{3k-2}\Phi_{3k-1}$ et l'on pourrait former le produit $BT_{3k-2}\Phi_{3k-1}$ et par suite le groupe BT_{3k-2} serait un tronçon n° $3k-2$ d'un ordre supérieur à celui du tronçon T_{3k-2}. Pour tout tronçon T_{3k-2} de l'ordre maximum on aura donc

$$T_{3k-2}\Phi_{3k-1} = \Phi_{3k-2} \quad (\text{gr. } \Phi_{3k-2})$$

et par suite
$$e_{3k-2} = g_{3k-2} - g_{3k-1}.$$

Le nombre des tronçons T_{3k-2} de l'ordre maximum sera égal au nombre des solutions de l'égalité

$$\Phi_{3k-1}X = \Phi_{3k-2} \quad (\text{gr. } \Phi_{3k-2})$$

c. à-d. à $s^{g_{3k-1}e_{3k-2}}$.

Pour un tronçon T_{3k-1} il y a deux cas à considérer.

1^er^ cas. On a
$$h^k_{k,\,k+1} \geqq h_k^{k,\,k+1}.$$

Dans ce cas je dis qu'il faut qu'on ait

$$T_{3k-1}Z^k_{k,\,k+1} = \Phi_{3k-1} \quad (\text{gr. } \Phi_{3k-1}).$$

En effet, le groupe Φ_{3k-1} étant le plus petit commun surgroupe de $T_{3k-1}Z^k_{k,\,k+1}$ et $T_{3k-1}Z_k^{k,\,k+1}$, l'ordre du plus grand commun sous-groupe de ces deux derniers groupes sera égal à

$$s^{e_{3k-1}+h^k_{k,\,k+1}+e_{3k-1}+h_k^{k,\,k+1}-g_{3k-1}} = s^{2e_{3k-1}+g_{3k}}$$

et par suite le groupe Φ_{3k-1} renfermera

$$s^{g_{3k-1}} - s^{e_{3k-1}+h^k_{k,\,k+1}} - s^{e_{3k-1}+h_k^{k,\,k+1}} + s^{2e_{3k-1}+g_{3k}}$$
$$= s^{2e_{3k-1}+g_{3k}}\left(s^{g_{3k-1}-h^k_{k,\,k+1}-e_{3k-1}} - 1\right)\left(s^{g_{3k-1}-h_k^{k,\,k+1}-e_{3k-1}} - 1\right) > 0$$

éléments ne faisant partie ni de $T_{3k-1}Z^k_{k,k+1}$ ni de $T_{3k-1}Z^{k,k+1}_k$. Or soit b un tel élément, le groupe B auquel il sert de base sera premier tant à $T_{3k-1}Z^k_{k,k+1}$ qu'à $T_{3k-1}Z^{k,k+1}_k$ et par suite le groupe BT_{3k-1} sera un tronçon n° $3k-1$ d'un ordre supérieur à celui de T_{3k-1}. Il en résulte qu'on doit avoir

$$e_{3k-1} + h^k_{k,k+1} = g_{3k-1}$$

et par suite $$T_{3k-1}Z^k_{k,k+1} = \Phi_{3k-1} \quad (\text{gr. } \Phi_{3k-1}).$$

2$^{\text{me}}$ cas. On a $$h^{k,k+1}_k \geqq h^k_{k,k+1}.$$

Dans ce cas on devra avoir

$$e_{3k-1} + h^{k,k+1}_k = g_{3k-1}$$

et $$T_{3k-1}Z^{k,k+1}_k = \Phi_{3k-1} \quad (\text{gr. } \Phi_{3k-1}).$$

Enfin quand $h^k_{k,k+1} = h^{k,k+1}_k$ les deux expressions donnent la même valeur de e_{3k-1} et l'on aura tant

$$T_{3k-1}Z^k_{k,k+1} = \Phi_{3k-1} \quad (\text{gr. } \Phi_{3k-1})$$

que $$T_{3k-1}Z^{k,k+1}_k = \Phi_{3k-1} \quad (\text{gr. } \Phi_{3k-1}).$$

Soit T_{3k-1} un tel tronçon de l'ordre maximum et

$$T_{3k-1} = T^1_{3k-1}T^2_{3k-1} \ldots\ldots T^{e_{3k-1}}_{3k-1}$$

une de ses décompositions en groupes simples, un facteur tel que T_{3k-1} de cette décomposition sera un sous-groupe de Φ_{3k-1} et sera premier tant au groupe $Z^k_{k,k+1}\Phi^1_{3k-1} \ldots\ldots \Phi^{l-1}_{3k-1}$ qu'au groupe $Z^{k,k+1}_k T^1_{3k-1}T^2_{3k-1} \ldots\ldots T^{l-1}_{3k-1}$. Inversement si l'on peut former une suite de e_{3k-1} sous-groupes simples de Φ_{3k-1}

$$\mathfrak{T}^1_{3k-1},\ \mathfrak{T}^2_{3k-1},\ \ldots\ldots \mathfrak{T}^{e_{3k-1}}_{3k-1}$$

tels que tout groupe $\mathfrak{T}^l_{3k-1}$ soit premier tant au groupe $Z^k_{k,k+1}\mathfrak{T}^1_{3k-1}\mathfrak{T}^2_{3k-1} \ldots \mathfrak{T}^{l-1}_{3k-1}$ qu'au groupe $Z^{k,k+1}_k\mathfrak{T}^1_{3k-1}\mathfrak{T}^2_{3k-1} \ldots\ldots \mathfrak{T}^{l-1}_{3k-1}$, tous les groupes de la suite pourront être réunis en un produit $\mathfrak{T}_{3k-1}$ qui sera un tronçon n° $3k-1$ de l'ordre maximum. Le groupe Φ_{3k-1} étant le plus petit commun surgroupe de

$$Z^k_{k,k+1}\mathfrak{T}^1_{3k-1}\mathfrak{T}^2_{3k-1} \ldots\ldots \mathfrak{T}^{l-1}_{3k-1} \quad \text{et} \quad Z^{k,k+1}_k\mathfrak{T}^1_{3k-1}\mathfrak{T}^2_{3k-2} \ldots\ldots \mathfrak{T}^{l-1}_{3k-1}$$

l'ordre du plus grand commun sous-groupe de ces deux derniers groupes sera égal à $$s^{2l-2+g_{3k}}.$$

Le nombre des éléments du groupe Φ_{3k-1} qui ne font partie ni de

$$Z^k_{k,k+1}\mathfrak{T}^1_{3k-1}\mathfrak{T}^2_{3k-1} \ldots\ldots \mathfrak{T}^{l-1}_{3k-1} \text{ ni de } Z^{k,k+1}_k\mathfrak{T}^1_{3k-1}\mathfrak{T}^2_{3k-1} \ldots\ldots \mathfrak{T}^{l-1}_{3k-1}$$

est donc égal à

$$s^{g_{3k-1}} - s^{l-1+h^k_{k,k+1}} - s^{l-1+h^{k,k+1}_k} + s^{2l-2+g_{3k}}$$
$$= s^{2l-2+g_{3k}}\left(s^{g_{3k-1}-h^k_{k,k+1}-l-1} - 1\right)\left(s^{g_{3k-1}-h^{k,k+1}_k-l-1} - 1\right).$$

Cette expression est toujours positive tant que $l-1<e_{3k-1}$ et par suite le groupe Φ_{3k-1} renfermera dans ce cas au moins un élément b ne faisant partie ni de $Z_{k,k+1}^{k}\mathfrak{T}_{3k-1}^{1}\mathfrak{T}_{3k-1}^{2}\ldots\ldots\mathfrak{T}_{3k-1}^{l-1}$ ni de $Z_{k}^{k,k+1}\mathfrak{T}_{3k-1}^{1}\mathfrak{T}_{3k-1}^{2}\ldots\ldots\mathfrak{T}_{3k-1}^{l-1}$. Si l'on désigne par B le groupe auquel b sert de base, ce groupe sera premier tant au groupe $Z_{k,k+1}^{k}\mathfrak{T}_{3k-1}^{1}\mathfrak{T}_{3k-1}^{2}\ldots\ldots\mathfrak{T}_{3k-1}^{l-1}$ qu'au groupe $Z_{k}^{k,k+1}\mathfrak{T}_{3k-1}^{1}\mathfrak{T}_{3k-1}^{2}\ldots\ldots\mathfrak{T}_{3k-1}^{l-1}$. On peut donc faire

$$\mathfrak{T}_{3k-1}^{l}=B \qquad (\text{gr. } \Phi_{3k-1}).$$

Le nombre des groupes $\mathfrak{T}_{3k-1}^{l}$ correspondant à une suite donnée

$$\mathfrak{T}_{3k-1}^{1},\ \mathfrak{T}_{3k-1}^{2}\ldots\ldots\mathfrak{T}_{3k-1}^{l-1}$$

est donc égal à

$$\frac{s^{g_{3k}+l-1}\left(s^{g_{3k-1}-h_{k,k+1}^{k}-l+1}-1\right)\left(s^{g_{3k-1}-h_{k}^{k,k+1}-l+1}-1\right)}{s-1}.$$

Le nombre de toutes les décompositions de tous les tronçons n° $3k-1$ de l'ordre maximum est donc égal à

$$\prod_{l=1}^{l=e_{3k-1}}\frac{s^{g_{3k}+l-1}\left(s^{g_{3k-1}-h_{k,k+1}^{k}-l+1}-1\right)\left(s^{g_{3k-1}-h_{k}^{k,k+1}-l+1}-1\right)}{s-1}.$$

Si l'on ne considère comme distinctes que les décompositions qui diffèrent au moins par un facteur, ce nombre se réduira à

$$\frac{1}{e_{3k-1}!}\prod_{l=1}^{l=e_{3k-1}}\frac{s^{g_{3k}+l-1}\left(s^{g_{3k-1}-h_{k,k+1}^{k}-l+1}-1\right)\left(s^{g_{3k-1}-h_{k}^{k,k+1}l+1}-1\right)}{s-1}.$$

Or le nombre des décompositions de tout tronçon n° $3k-1$ de l'ordre maximum en groupes simples est égal à

$$\frac{1}{e_{3k-1}!}\prod_{l=1}^{l=e_{3k-1}}\frac{s^{e_{3k-1}}-s^{l-1}}{s-1}=\frac{s^{\frac{e_{3k-1}^{2}-e_{3k-1}}{2}}}{e_{3k-1}!}\prod_{l=1}^{l=e_{3k-1}}\frac{s^{e_{3k-1}-l+1}-1}{s-1}.$$

Il en résulte que le nombre des tronçons n° $3k-1$ de l'ordre maximum est égal à

$$s^{e_{3k-1}g_{3k}}\prod_{l=1}^{l=e_{3k-1}}\frac{\left(s^{g_{3k-1}-h_{k,k+1}^{k}-l+1}-1\right)\left(s^{g_{3k-1}-h_{k}^{k,k+1}-l+1}-1\right)}{\left(s^{e_{3k-1}-l+1}-1\right)}$$

$$=s^{e_{3k-1}g_{3k}}\prod_{l=1}^{l=e_{3k-1}}\left(s^{g_{3k-1}-h^{(k)}-l+1}-1\right);$$

où

$$h^{(k)}=\frac{h_{k,k+1}^{k}\left[\frac{h_{k}^{k,k+1}}{h_{k,k+1}^{k}}\right]+h_{k}^{k,k+1}\left[\frac{h_{k,k+1}^{k}}{h_{k}^{k,k+1}}\right]}{\left[\frac{h_{k}^{k,k+1}}{h_{k,k+1}^{k}}\right]+\left[\frac{h_{k,k+1}^{k}}{h_{k}^{k,k+1}}\right]}.$$

Le nombre de toutes les décompositions en tronçons des sous-groupes principaux de rang 1 de tous les diviseurs suprêmes de $\underline{A}$ et $\overline{A}$ est donc égal à

$$\prod_{k=1}^{k=3\theta-2} s^{e_k g_{k+1}} \times \prod_{k=1}^{k=\theta-1} \prod_{l=1}^{l=e_{3k-1}} \left(s^{g_{3k-1}-h^{(k)}-l+1} - 1\right).$$

Le nombre des décompositions en tronçons du sous-groupe principal de rang 1 d'un diviseur D (de $\underline{A}$ et $\overline{A}$) ayant les groupes

$$D_1, D_2, \ldots . D_{3\theta-2}$$

pour groupes génériques et les nombres

$$i_1, i_2, \ldots . i_{3\theta-2}$$

pour nombres génériques, est facile à évaluer. En effet on a d'abord

$$T_{3\theta-2} = D_{3\theta-2} \quad (\text{gr. } D_{3\theta-2})$$

puis on peut prendre pour $T_{3\theta-3}$ toute solution de l'égalité

$$D_{3\theta-2} X = D_{3\theta-3} \quad (\text{gr. } D_{3\theta-3})$$

de même pour $T_{3\theta-4}$ on peut prendre toute solution de l'égalité

$$D_{3\theta-3} X = D_{3\theta-4} \quad (\text{gr. } D_{3\theta-4})$$

et ainsi de suite. Comme de cette manière on obtient toutes les décompositions de D_1 en tronçons, le nombre de ces décompositions sera égal à

$$s^{\sum\limits_{k=1}^{k=3\theta-2} e_k i_{k+1}}.$$

Le nombre des sous-groupes principaux de rang 1 des tous les diviseurs suprêmes de $\underline{A}$ et $\overline{A}$ est donc égal à

$$s^{\sum\limits_{k=1}^{k=3\theta-2} e_k (g_{k+1} - i_{k+1})} \cdot \prod_{k=1}^{k=\theta-1} \prod_{l=1}^{l=e_{3k-1}} \left(s^{g_{3k-1}-h^{(k)}-l+1} - 1\right)$$

où les nombres $$i_1, i_2, \ldots . i_{3\theta-2}$$

sont les nombres génériques de tout diviseur suprême.

En multipliant ce nombre par le nombre des diviseurs de $\underline{A}$ qui correspondent à un sous-groupe principal donné D_1, on aura le nombre des diviseurs suprêmes de $\underline{A}$ et $\overline{A}$.

Quant au nombre des diviseurs de $\underline{A}$ qui correspondent à un D_1 donné, il est égal à

$$s^{\sum\limits_{k=1}^{k=\theta} i_{3k+1}(g_{3k-2} - i_{3k-2})}.$$

18.

Soient, comme au paragraphe précédent, s^{e_k} l'ordre maximum que puisse atteindre un tronçon n° k de sous-groupe de diviseur (de $\underline{A}$ et $\overline{A}$), et

$$0 \leqq \varepsilon_k \leqq e_k.$$

Le nombre des tronçons n° $3k-1$ d'ordre $s^{\varepsilon_{3k-1}}$ sera égal à

$$\frac{1}{\varepsilon_{3k-1}!} \prod_{l=1}^{l=\varepsilon_{3k-1}} \frac{s^{g_{3k}+l-1}\left(s^{g_{3k-1}-h_{k,k+1}^{k}-l+1}-1\right)\left(s^{g_{3k-1}-h_k^{k,k+1}-l+1}-1\right)}{s-1}$$

$$:\frac{1}{\varepsilon_{3k-1}!} \prod_{l=1}^{l=\varepsilon_{3k-1}} \frac{s^{\varepsilon_{3k-1}}-s^{l-1}}{s-1}$$

$$= s^{\varepsilon_{3k-1} g_{3k}} \prod_{l=1}^{l=\varepsilon_{3k-1}} \frac{\left(s^{g_{3k-1}-h_{k,k+1}^{k}-l+1}-1\right)\left(s^{g_{3k-1}-h_k^{k,k+1}-l+1}-1\right)}{\left(s^{\varepsilon_{3k-1}-l+1}-1\right)}.$$

Le nombre des tronçons n° $3k-2$ d'ordre $s^{\varepsilon_{3k-2}}$ est égal à

$$\frac{1}{\varepsilon_{3k-2}!} \prod_{l=1}^{l=\varepsilon_{3k-2}} \frac{s^{g_{3k-2}}-s^{g_{3k-1}+l-1}}{s-1} : \frac{1}{\varepsilon_{3k-2}!} \prod_{l=1}^{l=\varepsilon_{3k-2}} \frac{s^{\varepsilon_{3k-2}}-s^{l-1}}{s-1}$$

$$= \prod_{l=1}^{l=\varepsilon_{3k-2}} \frac{s^{g_{3k-2}}-s^{g_{3k-1}+l-1}}{s^{\varepsilon_{3k-2}}-s^{l-1}}$$

$$= s^{g_{3k-1}\varepsilon_{3k-2}} \prod_{l=1}^{l=\varepsilon_{3k-2}} \frac{s^{g_{3k-2}-g_{3k-1}-l+1}-1}{s^{\varepsilon_{3k-2}-l+1}-1}.$$

Quant au nombre des tronçons n° $3k$, il est, comme nous le savons, toujours égal à un. Cela étant ainsi, le nombre de toutes décompositions en tronçons des sous-groupes principaux (de rang 1) de diviseur de $\underline{A}$ et $\overline{A}$ admettant les nombres ε, sera égal à

$$\prod_{k=1}^{k=3\theta-2} s^{\varepsilon_k g_{k+1}} \prod_{k=1}^{k=\theta-1} \prod_{l=1}^{l=\varepsilon_{3k-1}} \frac{\left(s^{g_{3k-1}-h_{k,k+1}^{k}-l+1}-1\right)\left(s^{g_{3k-1}-h_k^{k,k+1}-l+1}-1\right)}{\left(s^{\varepsilon_{3k-1}-l+1}-1\right)}$$

$$\times \prod_{k=1}^{k=\theta-1} \prod_{l=1}^{l=\varepsilon_{3k-2}} \frac{s^{g_{3k-2}-g_{3k-1}-l+1}-1}{s^{\varepsilon_{3k-2}-l+1}-1}.$$

Soient Δ un diviseur de $\underline{A}$ et $\overline{A}$ admettant les nombres ε,

$$\Delta_1,\ \Delta_2,\ \ldots\ \Delta_{3\theta-2}$$

ses groupes génériques et

$$\iota_1,\ \iota_2,\ \ldots\ \iota_{3\theta-2}$$

ses nombres génériques, on aura

$$\iota_k = \varepsilon_{3\theta-2} + \varepsilon_{3\theta-3} + \ldots\ \varepsilon_k.$$

Le nombre des décompositions de Δ_1 en tronçons sera égal à

$$s^{\sum_{k=1}^{k=3\theta-2} \varepsilon_k \iota_{k+1}}.$$

Le nombre des groupes Δ_1 correspondant à tous les diviseurs Δ admettant les nombres ε, sera par conséquent égal à

$$s^{\sum_{k=1}^{k=3\theta-2} \varepsilon_k (g_{k+1}-\iota_{k+1})}$$
$$\times \prod_{k=1}^{k=\theta-1} \prod_{l=1}^{l=\varepsilon_{3k-1}} \frac{\left(s^{g_{3k-1}-h^k_{k,k+1}-l+1}-1\right)\left(s^{g_{3k-1}-h_k^{k,k+1}-l+1}-1\right)}{\left(s^{\varepsilon_{3k-1}-l+1}-1\right)}$$
$$\times \prod_{k=1}^{k=\theta-1} \prod_{l=1}^{l=\varepsilon_{3k-2}} \frac{s^{g_{3k-2}-g_{3k-1}-l+1}-1}{s^{\varepsilon_{3k-2}-l+1}-1}.$$

Le nombre des diviseurs Δ (de $\underline{A}$ et $\overline{A}$) correspondant à un Δ_1 donné est égal au nombre des diviseurs de $\overline{\underline{A}}$ correspondant à ce même Δ_1. Or ce dernier nombre est égal à

$$\prod_{k=1}^{k=\theta} s^{\iota_{3k+1}(g_{3k-2}-\iota_{3k-2})}.$$

En multipliant le nombre des groupes Δ_1 par le nombre des groupes Δ qui correspondent à un Δ_1 donné, on aura le nombre des diviseurs Δ de $\underline{A}$ et $\overline{A}$ qui admettent des nombres ε satisfaisant aux conditions

$$0 \leqq \varepsilon_k \leqq e_k.$$

S'il s'agissait de déterminer le nombre des diviseurs Δ de $\underline{A}$ et $\overline{A}$ admettant des nombres caractéristiques ι_{3k-2} donnés d'avance, il faudrait poser

$$\iota_{3k-3} = \iota_{3k-2}$$

et puis intercaler des nombres ι_{3k-1} satisfaisant aux conditions

$$0 \leqq \iota_k - \iota_{k+1} \leqq e_k.$$

Le nombre des diviseurs Δ sera alors donné par une somme étendue à toutes les manières possibles d'intercaler les nombres ι_{3k-1}.

Faisons observer d'ailleurs que le nombre des Δ qui correspondent à un Δ_1 donné ne dépend que des nombres caractéristiques ι_{3k-2} de Δ et est indépendant des nombres ι_{3k-1}. La question se ramène donc à la recherche du nombre des groupes Δ_1.

Tout diviseur de $\underline{A}$ et $\overline{A}$ est d'ailleurs diviseur d'un diviseur suprême—car rien n'empêche de compléter ses tronçons jusqu'à ce qu'ils atteignent l'ordre maximum—mais il peut être diviseur de plusieurs diviseurs suprêmes en même temps.

19.

Tout nombre entier P n'est décomposable que d'une seule et unique manière en un produit de puissances de nombre premier premières entre elles

$$P = P_1 P_2 \dots P_\theta$$

si l'on convient de ne considérer comme distinctes que les décompositions qui diffèrent au moins par un facteur.

Le produit d'un diviseur de P_1 par un diviseur de P_2, par un diviseur de $P_3 \dots$ par un diviseur de P_θ, donne un diviseur de P et tous les diviseurs de P peuvent s'obtenir de cette manière. Chaque diviseur de P n'apparaîtra d'ailleurs qu'une seule fois.

Supposons maintenant qu'on ait deux nombres entiers P' et P'' décomposés en puissances de nombre premier

$$P' = P'_1 P'_2 \dots P'_{\theta_1},$$
$$P'' = P''_1 P''_2 \dots P''_{\theta_2}.$$

Soit d'une manière générale

$$P'''_k$$

une puissance de nombre premier divisant tant P' que P'' et figurant explicitement au moins dans une des deux décompositions, j'écris la suite complète de telles puissances

$$P'''_1, P'''_2, \dots P'''_{\theta_3}$$

et je pose

$$P''' = P'''_1 P'''_2 \dots P'''_{\theta_3}.$$

Le produit d'un diviseur de P'''_1 par un diviseur de $P'''_2, \dots$ par un diviseur de P_{θ_3}, donnera un diviseur commun de P' et de P'' et tous les diviseurs communs de ces deux nombres pourront s'obtenir de cette manière. Chaque diviseur n'apparaîtra d'ailleurs qu'une seule fois. En particulier le nombre P''' qui est plus grand que tout autre diviseur commun de P' et P'', est dit le plus grand commun diviseur de ces deux nombres. Tous les diviseurs communs de P' et P'' sont donc des diviseurs de P''' et inversement.

Tout groupe Ξ_θ est décomposable en un produit de groupes isoclones de rangs différents

$$\Xi_\theta = T_\theta T_{\theta-1} \dots T_1$$

où T_k désigne un groupe isoclone de rang k. Soit $s^{k(m_k - m_{k+1})}$ l'ordre du groupe T_k et $m_{\theta+1} = 0$, les nombres

$$m_1, m_2, \dots m_\theta$$

sont les nombres caractéristiques de Ξ_θ. Si

$$\Xi_\theta = T'_{\theta'} T'_{\theta'-1} \ldots . T'_1$$

est une autre décomposition de Ξ_θ en groupes isoclones de rangs différents et $s^{k(m'_k - m'_{k+1})}$ l'ordre du groupe T'_k de rang k de cette décomposition, on aura, pour toute valeur de k,

$$m_k = m'_k$$

et

$$\theta = \theta'.$$

Inversement si T'_k désigne un diviseur isoclone de Ξ_θ de rang k et d'ordre $s^{k(m_k - m_{k+1})}$ on aura

$$\Xi_\theta = \prod_{k=1}^{k=\theta} T'_k.$$

Soit D_k un diviseur quelconque de T'_k sans excluse le groupe-unité auquel le groupe T'_k peut être égal lui-même, on pourra poser

$$D = \prod_{k=1}^{k=\theta} D_k$$

et D sera un diviseur de Ξ_θ. En donnant à D_k toutes les valeurs possibles, on aura tous les diviseurs D de Ξ_θ et chaque diviseur apparaîtra sous toutes ses formes, une fois sous chaque forme. Le nombre des diviseurs D admettant des nombres caractéristiques n_k donnés d'avance, est une fonction des nombres s, m_k et n_k.

Soient maintenant A et B deux sous-groupes de Ξ_θ et m_k et n_k leurs nombres caractéristiques, on peut réunir dans une seule collection n° k tous les diviseurs isoclones de rang k communs à A et à B. Alors le produit d'un groupe de la collection n° 1, par un groupe de la collection n° 2 et ainsi de suite jusqu'à la dernière collection n° η par exemple, donnera un diviseur commun de A et B et tous les diviseurs communs de A et B pourront s'obtenir de cette manière et de sorte que chaque diviseur apparaîtra sous toutes ses formes, une fois sous chaque forme. L'inconvénient de cette manière de procéder consiste en ce que le nombre des diviseurs communs D de A et B admettant des nombres caractéristiques r_k donnés d'avance, n'est pas une fonction des nombres s, m_k, n_k, r_k seuls.

Si, dans chaque collection, on prend un groupe de l'ordre maximum, on obtiendra un diviseur de A et B de l'ordre maximum—un diviseur suprême. Mais un tel diviseur n'est pas unique. Il est vrai encore que tout diviseur de A et B est un diviseur d'un diviseur suprême et inversement tout diviseur d'un

diviseur suprême est un diviseur commun de A et B. Mais un diviseur commun de A et B peut être diviseur de plusieurs diviseurs suprêmes en même temps.

Nous avons introduit le plus grand commun sous-groupe C de A et B et il s'est trouvé que tout diviseur commun de A et B est un diviseur de C, mais l'inverse n'a pas lieu. Soient c_k les nombres caractéristiques de C, il nous a fallu intercaler entre chaque couple de nombres c_k et c_{k+1} quatre nombres

$$h^{k+1,k}_{k+1,k},\ h^{k,k+1}_{k},\ h^{k}_{k,k+1},\ h^{k,k+1}_{k,k+1}$$

satisfaisant aux conditions

$$h^{k+1,k}_{k+1,k} + h^{k,k+1}_{k,k+1} = h^{k,k+1}_{k} + h^{k}_{k,k+1},$$
$$c_{k+1} \leqq h^{k+1,k}_{k+1,k} \leqq h^{k,k+1}_{k} \leqq h^{k}_{k,k+1} \leqq h^{k,k+1}_{k,k+1} \leqq c_k.$$

Le nombre des diviseurs communs D de A et B admettant des nombres r_k donnés d'avance, est une fonction des nombres s, c, h, r seuls.

20.

Soit A un sous-groupe quelconque de Ξ_θ, A_k le plus grand commun sous-groupe de A et H_k et s^{a_k} l'ordre de A_k. Désignons par

$$\Phi_1, \Phi_2, \ldots. \Phi_\theta$$

les sous-groupes caractéristiques et par

$$\alpha_1, \alpha_2, \ldots. \alpha_\theta$$

les nombres caractéristiques de A.

On aura, comme au §17,

$$A_1 = \Phi_1$$

et Φ_k sera toujours un sous-groupe de A_k. On aura donc

$$\alpha_k \leqq a_k.$$

Cela étant ainsi, posons

$$T_\theta = A_\theta \qquad (\text{gr. } \Xi_\theta)$$

et

$$A_k = T_k A_{k+1} \qquad (\text{gr. } \Xi_\theta)$$

on aura

$$A_1 = T_1 T_2 \ldots. T_\theta$$

où les T seront les *tronçons* (gr. Ξ_θ) de A_1. Le nombre des décompositions de A_1 en tronçons (gr. Ξ_θ) est égal à

$$\prod_{k=1}^{k=\theta} s^{(\alpha_k - \alpha_{k+1})\, a_{k+1}}.$$

Je dis maintenant que T_k est premier à H_{k+1}. En effet, le plus grand commun sous-groupe de A et H_{k+1} étant égal à A_{k+1}, le plus grand commun sous-groupe

de $A_k = T_k A_{k+1}$ et H_{k+1} sera aussi égal à A_{k+1} et par conséquent T_k est premier à H_{k+1}. Désignons maintenant par Ψ_k le plus petit commun surgroupe de H_{k+1} et A_k, Ψ_k sera aussi un surgroupe de H_{k+1} et T_k et par suite de $H_{k+1}T_k$. Or $H_{k+1}T_k$ étant un surgroupe tant de H_{k+1} que de A_k, on aura

$$\Psi_k = H_{k+1}T_k \quad (\text{gr. } \Xi_\theta)$$

c. à-d. que toute solution de l'égalité

$$A_k = A_{k+1}X \quad (\text{gr. } A_k)$$

satisfait à l'égalité $\quad \Psi_k = H_{k+1}X \quad (\text{gr. } \Psi_k).$

Je désignerai maintenant sous le nom de tronçon n° k (gr. Ξ_θ) tout sous-groupe $\mathfrak{T}_k$ de H_k premier à H_{k+1}. Si l'on prend un tronçon n° 1, un tronçon n° 2, ... un tronçon n° θ, on obtient un système de tronçons (gr. Ξ_θ)

$$\mathfrak{T}_1,\ \mathfrak{T}_2,\ \ldots .\ \mathfrak{T}_\theta.$$

Tous les tronçons d'un système peuvent être réunis en un produit $\mathfrak{T}_1, \mathfrak{T}_2, \ldots . \mathfrak{T}_\theta$ qui sera naturellement un sous-groupe de Ξ_θ et inversement tout sous-groupe de Ξ_θ peut être décomposé en un produit de tronçons (gr. Ξ_θ) formant système. Soit maintenant C un multiple de A de sorte qu'on ait

$$AB = C \quad (\text{gr. } C)$$

par exemple et soit C_k le plus grand commun sous-groupe de C et H_k. Il est clair que le plus grand commun sous-groupe de C_k et A sera égal à A_k. Désignons maintenant par Z_k le plus petit commun surgroupe de C_{k+1} et A_k, on aura encore

$$C_{k+1}T_k = Z_k \quad (\text{gr. } Z_k).$$

Le groupe Z_k sera d'ailleurs un sous-groupe de C_k. Nous poserons

$$Z_\theta = A_\theta = T_\theta,$$

$$Z_k U_k = C_k \quad (\text{gr. } C_k)$$

on aura $\quad C = T_\theta U_\theta T_{\theta-1} U_{\theta-1} \ldots . T_1 U_1.$

Je dis que U_k est premier à Ψ_k. En effet, si U_k n'était pas premier à Ψ_k, $T_k U_k$ ne serait pas premier à H_{k+1} et ces deux derniers groupes auraient en commun un élément a différent de l'élément-unité. Or l'élément a fait partie de C et de H_{k+1} et par suite de C_{k+1} de sorte que C_k et $T_k U_k$ ne seraient pas premiers et $C_{k+1}T_k = Z_k$ et U_k non plus, contrairement à la supposition que U_k est une solution de l'égalité $\quad Z_k X = C_k \quad (\text{gr. } C_k).$

Je désignerai par Θ_k le plus petit commun surgroupe de H_{k+1} et C_k; on aura

$$H_{k+1} V_k = \Theta_k \quad (\text{gr. } \Theta_k).$$

Si l'on désigne par V_k un tronçon n° k (gr. Ξ_θ) du groupe C_1. Le groupe $T_k U_k$ étant aussi un tronçon n° k (gr. Ξ_θ) du groupe C_1, on aura

$$H_{k+1} T_k U_k = \Theta_k \quad (\text{gr. } \Theta_k)$$

d'où

$$\Psi_k U_k = \Theta_k \quad (\text{gr. } \Theta_k).$$

Je pose maintenant

$$B_1 = U_\theta U_{\theta-1} \dots \dots U_1;$$

on aura

$$A_1 B_1 = C_1 \quad (\text{gr. } C)$$

où B_1 admet une décomposition en tronçons (gr. Ξ_θ) tels que le tronçon n° k vérifie la condition d'être premier non seulement à H_{k+1}, mais aussi à $\Psi_k = H_{k+1} T_k$. Je dis que toute décomposition de B_1 en tronçons (gr. Ξ_θ)

$$B_1 = U'_\theta U'_{\theta-1} \dots \dots U'_1$$

jouira de la même propriété. En effet, si U'_k qui est premier à $U'_\theta U'_{\theta-1} \dots U'_{k+1}$ avait en commun avec Ψ_k un élément a différent de l'élément-unité, les groupes Ψ_k et B_1 auraient un commun sous-groupe G d'un ordre supérieur à celui du groupe $U'_\theta U'_{\theta-1} \dots U'_{k+1}$ et par suite aussi à celui du groupe $U_\theta U_{\theta-1} \dots U_{k+1}$, ce qui est impossible. Si donc C est un multiple de A, on peut trouver un facteur complémentaire B_1 de A_1 par rapport à C_1 d'une nature telle que dans toute décomposition de B_1 en tronçons (gr. Ξ_θ)

$$B_1 = U_\theta U_{\theta-1} \dots . U_1,$$

U_k sera premier à Ψ_k et à plus forte raison à Z_k de sorte qu'on aura

$$Z_k U_k = C_k \quad (\text{gr. } C_k).$$

Un tel facteur complémentaire B_1 portera le nom de facteur complémentaire convenable de A_1 par rapport à C_1.

Le nombre des tronçons U_k est donné par le nombre des solutions de l'égalité

$$Z_k X = C_k \quad (\text{gr. } C_k)$$

qui est égal à

$$s^{(b_k - b_{k+1})\{c_{k+1} + (a_k - a_{k+1})\}}$$

où

$$b_k = c_k - a_k.$$

Le nombre de toutes les décompositions de tous les groupes B_1 est donc égal à

$$\prod_{k=1}^{k=\theta} s^{(b_k - b_{k+1})\{c_{k+1} + (a_k - a_{k+1})\}}.$$

Quant au nombre des décompositions d'un groupe B_1 en tronçons (gr. Ξ_θ), il est égal à

$$\prod_{k=1}^{k=\theta} s^{b_{k+1}(b_k - b_{k+1})}.$$

Le nombre des groupes B_1 est donc égal à

$$\prod_{k=1}^{k=\theta} s^{a_k(b_k - b_{k+1})}.$$

Cela nous donne le nombre des solutions de l'égalité

$$A_1 X_1 = C_1 \quad (\text{gr. } C)$$

où X_1 doit être un facteur complémentaire convenable de A_1 par rapport à C_1. Le nombre des tronçons U_k pouvant servir à former un facteur complémentaire convenable B_1 de A_1 non seulement par rapport à C_1, mais aussi par rapport à tout multiple de A_1 admettant les mêmes nombres c, est égal à

$$\frac{1}{(b_k - b_{k+1})!} \prod_{l=1}^{l=b_k - b_{k+1}} \frac{s^{m_k} - s^{m_{k+1} + a_k - a_{k+1} + l - 1}}{s - 1}$$

divisé par

$$\frac{1}{(b_k - b_{k+1})!} \prod_{l=1}^{l=b_k - b_{k+1}} \frac{s^{b_k - b_{k+1}} - s^{l-1}}{s - 1}.$$

En effet, les groupes U_k ne sont pas autre chose que des sous-groupes de H_k d'ordre $s^{b_k - b_{k+1}}$ et premier à Ψ_k. Réduction faite, on trouve l'expression

$$s^{(b_k - b_{k+1})(m_{k+1} + a_k - a_{k+1})} \prod_{l=1}^{l=b_k - b_{k+1}} \frac{s^{m_k - m_{k+1} - a_k + a_{k+1} - l + 1} - 1}{s^{b_k - b_{k+1} - l + 1} - 1}.$$

Le nombre des groupes B_1 est donc égal à

$$\prod_{k=1}^{k=\theta} s^{(b_k - b_{k+1})(m_{k+1} + a_k - a_{k+1})}$$
$$\times \prod_{k=1}^{k=\theta} \prod_{l=1}^{l=b_k - b_{k+1}} \frac{s^{m_k - m_{k+1} - a_k + a_{k+1} - l + 1} - 1}{s^{b_k - b_{k+1} - l + 1} - 1}.$$

Le nombre des groupes C_1 admettant les mêmes nombres c est donc égal à

$$\prod_{k=1}^{k=\theta} s^{(b_k - b_{k+1})(m_{k+1} - a_{k+1})}$$
$$\times \prod_{k=1}^{k=\theta} \prod_{l=1}^{l=b_k - b_{k+1}} \frac{s^{m_k - m_{k+1} - a_k + a_{k+1} - l + 1} - 1}{s^{b_k - b_{k+1} - l + 1} - 1}.$$

Soit C un multiple de A admettant C_1 pour sous-groupe principal de rang 1; désignons par

$$\Gamma_1, \Gamma_2, \ldots . \Gamma_\theta$$

les groupes caractéristiques et par

$$\gamma_1, \gamma_2, \ldots . \gamma_\theta$$

les nombres caractéristiques de C. On aura

$$a_k \leqq \gamma_k \leqq c_k \quad (k = 1, 2, \ldots . \theta).$$

Posons $$\beta_k = \gamma_k - a_k \quad (k = 1, 2, \ldots . \theta)$$

les nombres β seront les nombres caractéristiques de toute groupe B pour lequel on a

$$AB = C \quad (\text{gr. } C).$$

Soit B_1 un facteur complémentaire convenable de A_1 par rapport à C_1, si l'on désigne par B_k le plus grand commun sous-groupe de B_1 et H_k et par s^{b_k} l'ordre de B_k, on aura

$$b_{k+1} \leqq b_k \quad (k = 1, 2, \ldots . \theta),$$
$$a_k + b_k = c_k \quad (k = 1, 2, \ldots . \theta).$$

Prenons maintenant une décomposition convenable (C_1) de C_1 (par rapport à C) telle que tous les facteurs d'une décomposition convenable de A_1 (par rapport à A) y figurent explicitement et désignons par $(C_1)'$ ce que devient (C_1) quand on y remplace chaque facteur de (A_1) par un de ses émanants de l'ordre maximum (dans le groupe A). Si dans $(C_1)'$ on remplace de toutes les manières possibles, chaque facteur d'ordre s (ne faisant pas partie de (A_1)) par un de ses émanants (dans le groupe Ξ_θ) d'un ordre égal à la portée du dit facteur dans le groupe C, on obtiendra tous les multiples C de A admettant les groupes caractéristiques Γ et ayant par conséquent le groupe C_1 pour sous-groupe principal de rang 1. Un tel groupe C apparaîtra d'ailleurs autant de fois qu'il existe d'expressions $(C_1)''$ qu'on peut obtenir de l'expression $(C_1)'$ en y remplaçant chaque facteur d'ordre s (ne faisant pas partie de (A_1)) par un de ses émanants de l'ordre maximum dans le groupe C. Le nombre de tels groupes C que nous considérerons comme constituant une classe C, est donc égal à

$$\prod_{k=1}^{k=\theta} s^{(\beta_k - \beta_{k+1})\left\{\sum_{u=1}^{u=k-1} (m_u - \gamma_u)\right\}}.$$

Les groupes $$\Gamma_1, \Gamma_2, \ldots . \Gamma_\theta$$

jouissent des propriétés suivantes. Le groupe Γ_k est d'ordre s^{γ_k}; il est surgroupe tant du groupe Φ_k que du groupe Γ_{k+1} et sous-groupe de C_k. Comme $\gamma_1 = c_1$, le groupe Γ_1 est d'ailleurs égal à C_1. Inversement si l'on forme une suite de groupes

$$\Gamma_1', \Gamma_2', \ldots . \Gamma_\theta'$$

tels que Γ'_k soit d'ordre s^{γ_k} et qu'il soit surgroupe tant de Φ_k que de Γ'_{k+1} et sous-groupe de C_k, il existera un multiple C' de A admettant les groupes Γ' comme sous-groupes caractéristiques et ayant par conséquent les mêmes nombres caractéristiques que C et le même sous-groupe principal de rang 1 C_1. En effet, il suffit de décomposer le groupe Γ'_1 de manière que la dite décomposition renferme explicitement une décomposition convenable de A_1 (par rapport à A) aussi bien que des décompositions des groupes

$$\Gamma'_2, \Gamma'_3, \ldots . \Gamma'_\theta.$$

Si l'on remplace dans une telle décomposition (Γ'_1) de Γ'_1 tout groupe qui fait partie de A_1 par un de ses émanants de l'ordre maximum (dans le groupe A) et tout groupe qui fait partie de Γ'_k sans faire partie de Φ_k ni de Γ'_{k+1}, par un de ses émanants de l'ordre s^k (dans le groupe Ξ_θ), on obtiendra bien un groupe C' ayant les propriétés requises. Les groupes

$$\Gamma'_1, \Gamma'_2, \ldots . \Gamma'_\theta$$

nous donneront ainsi une classe C' de multiples de A admettant les nombres caractéristiques γ et ayant C_1 pour sous-groupe principal de rang 1. La classe C' renferme naturellement le même nombre d'individus que la classe C. Le nombre des classes C, C', etc., est donc égal au nombre de systèmes de groupes

$$\Gamma_1, \Gamma_2, \ldots . \Gamma_\theta.$$

Le nombre des groupes Γ_θ est égal au nombre des multiples de Φ_θ qui sont d'ordre s^{γ_θ} et qui sont en même temps sous-groupes de C_θ. Pour obtenir un tel multiple Γ_θ de Φ_θ, je multiplie Φ_θ par un sous-groupe $\mathfrak{T}_1$ de C_θ d'ordre s et premier à Φ_θ. Il y aura, comme nous le savons,

$$\frac{s^{c_\theta} - s^{a_\theta}}{s - 1}$$

de tels groupes $\mathfrak{T}_1$. Multiplions $\Phi_\theta \mathfrak{T}_1$ par un sous-groupe $\mathfrak{T}_2$ de C_θ d'ordre s et premier à $\Phi_\theta \mathfrak{T}_1$. Il y en aura

$$\frac{s^{c_\theta} - s^{a_\theta + 1}}{s - 1}$$

de tels groupes $\mathfrak{T}_2$. En continuant de la même manière, on finira par obtenir un tel Γ_θ sous la forme

$$\Phi_\theta \mathfrak{T}_1 \mathfrak{T}_2 \ldots . \mathfrak{T}_{\gamma_\theta - a_\theta}.$$

Le nombre des produits explicites

35 $$\mathfrak{T}_1 \mathfrak{T}_2 \ldots . \mathfrak{T}_{\gamma_\theta - a_\theta}$$

sera égal à
$$\frac{1}{(\gamma_\theta-\alpha_\theta)!}\prod_{l=1}^{l=\gamma_\theta-\alpha_\theta}\frac{s^{\sigma_\theta}-s^{\alpha_\theta+l-1}}{s-1}$$

si l'on ne considère pas comme distincts les produits qui ne diffèrent que par l'ordre de leurs facteurs. Le nombre des produits implicites

$$\mathfrak{T}_1\mathfrak{T}_2\ldots\mathfrak{T}_{\gamma_\theta-\alpha_\theta}$$

différents s'obtiendra en divisant le nombre précédent

$$\frac{1}{(\gamma_\theta-\alpha_\theta)!}\prod_{l=1}^{l=\gamma_\theta-\alpha_\theta}\frac{s^{\sigma_\theta}-s^{\alpha_\theta+l-1}}{s-1}$$

par
$$\frac{1}{(\gamma_\theta-\alpha_\theta)!}\prod_{l=1}^{l=\gamma_\theta-\alpha_\theta}\frac{s^{\gamma_\theta-\alpha_\theta}-s^{l-1}}{s-1}$$

ce qui donne
$$s^{\alpha_\theta(\gamma_\theta-\alpha_\theta)}\prod_{l=1}^{l=\gamma_\theta-\alpha_\theta}\frac{s^{\sigma_\theta-\alpha_\theta-l+1}-1}{s^{\gamma_\theta-\alpha_\theta-l+1}-1}.$$

Le nombre des solutions d'une égalité telle que

$$\Phi_\theta X=\Gamma_\theta \quad (\text{gr. } \Gamma_\theta)$$

étant égal à
$$s^{\alpha_\theta(\gamma_\theta-\alpha_\theta)}$$

le nombre des groupes Γ_θ sera égal à

$$\prod_{l=1}^{l=\gamma_\theta-\alpha_\theta}\frac{s^{\sigma_\theta-\alpha_\theta-l+1}-1}{s^{\gamma_\theta-\alpha_\theta-l+1}-1}.$$

Si les groupes
$$\Gamma_\theta,\ \Gamma_{\theta-1},\ldots\Gamma_{k+1}$$

sont déjà fixés, le nombre des groupes Γ_k qu'on peut leur associer, est égal au nombre des sous-groupes de C_k qui sont d'ordre s^{γ_k} et qui sont multiples de Γ_{k+1} et de Φ_k, ou, ce qui revient au même, qui sont multiples de leur plus petit commun surgroupe que nous désignerons par Λ_k. Le plus grand commun sous-groupe de Φ_k et Γ_{k+1} étant égal à Φ_{k+1}, l'ordre du groupe Λ_k sera égal à

$$s^{\alpha_k+\gamma_{k+1}-\alpha_{k+1}}.$$

Le nombre des groupes Γ_k est donc égal à

$$\prod_{l=1}^{l=\gamma_k-\gamma_{k+1}-\alpha_k+\alpha_{k+1}}\frac{s^{\sigma_k-\gamma_{k+1}-\alpha_k+\alpha_{k+1}-l+1}-1}{s^{\gamma_k-\gamma_{k+1}-\alpha_k+\alpha_{k+1}-l+1}-1}.$$

Le nombre des systèmes de groupes

$$\Gamma_1,\ \Gamma_2,\ldots\Gamma_\theta$$

est donc égal à

$$\prod_{k=1}^{k=\theta}\ \prod_{l=1}^{l=\gamma_k-\gamma_{k+1}-a_k+a_{k+1}} \frac{g^{c_k-\gamma_{k+1}-a_k+a_{k+1}-l+1}-1}{g^{\gamma_k-\gamma_{k+1}-a_k+a_{k+1}-l+1}-1}.$$

Le nombre des multiples C de A admettant les nombres caractéristiques γ et ayant pour sous-groupe principal de rang 1 un groupe C_1 donné (admettant les nombres c) est donc égal à

$$\prod_{k=1}^{k=\theta} g^{(\beta_k-\beta_{k+1})\left\{\sum\limits_{u=1}^{u=k-1}(m_u-\gamma_u)\right.}$$
$$\times \prod_{k=1}^{k=\theta}\ \prod_{l=1}^{l=\beta_k-\beta_{k+1}} \frac{g^{c_k-\gamma_{k+1}-a_k+a_{k+1}-l+1}-1}{g^{\gamma_k-\gamma_{k+1}-a_k+a_{k+1}-l+1}-1}.$$

Enfin le nombre des multiples C de A admettant les nombres c et les nombres caractéristiques γ est égal à

$$\prod_{k=1}^{k=\theta} g^{(b_k-b_{k+1})(m_{k+1}-a_{k+1})}$$
$$\times \prod_{k=1}^{k=\theta} g^{(\beta_k-\beta_{k+1})\left\{\sum\limits_{u=1}^{u=k-1}(m_u-\gamma_u)\right.}$$
$$\times \prod_{k=1}^{k=\theta}\ \prod_{l=1}^{l=b_k-b_{k+1}} \frac{g^{m_k-m_{k+1}-a_k+a_{k+1}-l+1}-1}{g^{b_k-b_{k+1}-l+1}-1}$$
$$\times \prod_{k=1}^{k=\theta}\ \prod_{l=1}^{l=\beta_k-\beta_{k+1}} \frac{g^{c_k-\gamma_{k+1}-a_k+a_{k+1}-l+1}-1}{g^{\gamma_k-\gamma_{k+1}-a_k+a_{k+1}-l+1}-1}.$$

Les nombres a peuvent être aussi définis comme étant les nombres caractéristiques de tout diviseur de Ξ_θ ayant A_1 pour sous-groupe principal de rang 1. Pour obtenir un tel diviseur de Ξ_θ, il suffit de décomposer A_1 de manière que des décompositions des groupes

$$A_2,\ A_3,\ \ldots\ A_\theta$$

figurent explicitement dans la décomposition de A_1 et de remplacer dans cette décomposition de A_1 chaque facteur par un de ses émanants de l'ordre maximum dans le groupe Ξ_θ. De même les nombres c peuvent être définis comme étant les nombres caractéristiques de tout diviseur de Ξ_θ ayant C_1 pour sous-groupe principal de rang 1. Soit $\mathfrak{C}$ un tel diviseur de Ξ_θ, si dans une décomposition de A_1 telle que nous l'avons définie plus haut, on remplace chaque facteur par un de ses émanants de l'ordre maximum dans le groupe $\mathfrak{C}$, on obtient un diviseur $\mathfrak{A}$ de $\mathfrak{C}$

qui sera aussi un diviseur de Ξ_θ. Les nombres c sont donc assujettis aux conditions

$$(a_k - a_{k+1}) \leqq (c_k - c_{k+1}) \leqq (m_k - m_{k+1}) \qquad (k = 1, 2, \ldots . \theta)$$

où

$$a_{\theta+1} = c_{\theta+1} = m_{\theta+1} = 0.$$

Les nombres γ sont assujettis aux conditions

$$(\alpha_k - \alpha_{k+1}) \leqq (\gamma_k - \gamma_{k+1}) \qquad (k = 1, 2, \ldots . \theta),$$

$$\gamma_k \leqq c_k \qquad (k = 2, 3, \ldots . \theta),$$

$$\gamma_1 = c_1.$$

21.

Au point où nous sommes parvenus, l'exposition du développement ultérieur de la théorie des groupes Ξ_θ, n'offre d'autres difficultés que celles d'élocution et de notation. Je ne m'y attarde donc pas, et je passe rapidement à l'exposition du *théorème fondamental de Gauss* qui nous permettra de ramener la théorie des groupes eulériens quelconques à celle des groupes Ξ_θ.

22.

Soit Ω un groupe eulérien quelconque d'ordre ω et

$$s_1, s_2, \ldots . s_n$$

les nombres premiers par rapport auxquels le groupe Ω est d'un rang supérieur à zéro. Comme l'ordre du groupe Ω est fini, le nombre des nombres premiers s sera aussi fini. Nous désignerons, d'une manière générale, par θ_k le rang du groupe Ω par rapport au nombre premier s_k, par Ξ_{θ_k} le groupe formé par l'ensemble des solutions de l'égalité

$$x^{s^{\theta_k}} = 1 \quad (\text{gr. } \Omega)$$

et par $s_k^{\nu_k}$ son ordre.

Le groupe Ξ_{θ_1} ne contient que des éléments appartenant aux exposants

$$1, s_1, s_1^2, \ldots . s_1^{\theta_1}$$

et le groupe Ξ_{θ_2} que des éléments appartenant aux exposants

$$1, s_2, s_2^2, \ldots . s_2^{\theta_2}$$

ces deux groupes n'ont donc de commun que l'élément-unité et on pourra les

combiner de manière à obtenir un groupe $\Xi_{\theta_1}\Xi_{\theta_2}$ d'ordre $s_1^{w_1}s_2^{w_2}$. Ce dernier groupe ne contiendra que des éléments appartenant à des exposants de la forme

$$s_1^{t_1}s_2^{t_2} \quad (t_1 = 0, 1, \ldots . \theta_1; \; t_2 = 0, 1, \ldots . \theta_2)$$

et n'aura donc de commun avec le groupe Ξ_{θ_3} que l'élément-unité. On peut donc combiner les groupes $\Xi_{\theta_1}\Xi_{\theta_2}$ et Ξ_{θ_3} de manière à obtenir un groupe $\Xi_{\theta_1}\Xi_{\theta_2}\Xi_{\theta_3}$ d'ordre $s^{w_1}s_2^{w_2}s_3^{w_3}$. En continuant de la même manière on finira par obtenir un groupe

$$\Psi = \Xi_{\theta_1}\Xi_{\theta_2} \ldots . \Xi_{\theta n}$$

d'ordre $s_1^{w_1}s_2^{w_2} \ldots . s_n^{w_n}$, lequel groupe sera évidemment un sous-groupe de Ω. Or soit a un élément quelconque du groupe Ω appartenant à un exposant

$$u = p_1^{\pi_1}p_2^{\pi_2} \ldots . p_s^{\pi_s}$$

où $p_1, p_2, \ldots p_s$ sont des nombres premiers différents. Déterminons les nombres

$$\alpha_1, \alpha_2, \ldots . \alpha_s$$

de manière qu'on ait*

$$\alpha_1 \equiv 1 \;(\text{mod. } p_1^{\pi_1}) \quad \alpha_1 \equiv 0 \left(\text{mod. } \frac{u}{p_1^{\pi_1}}\right),$$
$$\alpha_2 \equiv 1 \;(\text{mod. } p_2^{\pi_2}) \quad \alpha_2 \equiv 0 \left(\text{mod. } \frac{u}{p_2^{\pi_2}}\right),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\alpha_s \equiv 1 \;(\text{mod. } p_s^{\pi_s}) \quad \alpha_s \equiv 0 \left(\text{mod. } \frac{u}{p_s^{\pi_s}}\right)$$

et posons
$$a^{\alpha_1} = b_1,$$
$$a^{\alpha_2} = b_2,$$
$$\ldots\ldots$$
$$\ldots\ldots$$
$$a^{\alpha_s} = b_s.$$

Un élément tel que b_k appartiendra à l'exposant $p_k^{\pi_k}$, car on aura

$$b_k^{p_k^{\pi_k}} = a^{\alpha_k p_k^{\pi_k}} = 1 \qquad (\text{gr. } \Omega)$$

tandis que
$$b_k^{p_k^{\pi_k - 1}} = a^{\alpha_k p_k^{\pi_k - 1}}$$

ne pourra être égal à l'unité, l'exposant $\alpha_k p_k^{\pi_k - 1}$ n'étant pas divisible par u.

Les nombres premiers

$$p_1, p_2, \ldots . p_s$$

* Disquisitiones arithmeticae, art. 36 et 81.

font donc nécessairement partie de la suite

$$s_1,\ s_2,\ \ldots .\ s_n$$

et l'on aura, par exemple,

$$p_1 = s_{v_1} \quad p_2 = s_{v_2} \ \ldots .\ p_z = s_{v_z}$$

où

$$v_1,\ v_2,\ \ldots .\ v_z$$

sont des nombres de la suite

$$1,\ 2,\ \ldots .\ n.$$

Si $z < n$, désignons par

$$v_{z+1},\ v_{z+2},\ \ldots .\ v_n$$

les nombres de la suite

$$1,\ 2,\ \ldots .\ n$$

qui manquent dans la suite

$$v_1,\ v_2,\ \ldots .\ v_z$$

pris dans un ordre quelconque et posons

$$b_h = 1 \qquad (\text{gr. } \Omega)$$

pour toute valeur de h satisfaisant à la condition

$$z < h \leqq n$$

on aura

$$a = b_1 b_2 \ldots .\ b_n \quad (\text{gr. } \Omega)$$

où les éléments $b_1,\ b_2,\ \ldots .\ b_n$ font partie respectivement des groupes

$$\Xi_{\theta_{v_1}},\ \Xi_{\theta_{v_2}},\ \ldots .\ \Xi_{\theta_{v_n}}.$$

L'élément a fait donc partie du groupe Ψ et par suite Ω est un sous-groupe de Ψ. On aura donc

$$\Omega = \Psi = \Xi_{\theta_1} \Xi_{\theta_2} \ldots .\ \Xi_{\theta_n}$$

et

$$\omega = s_1^{w_1} s_2^{w_2} \ldots .\ s_n^{w_n}.$$

C'est le *théorème fondamental de Gauss** qui peut être énoncé de la manière suivante.

Soit Ω un groupe eulérien quelconque et

$$\omega = s_1^{\sigma_1} s_2^{\sigma_2} \ldots .\ s_n^{\sigma_n}$$

son ordre, où $s_1,\ s_2,\ \ldots .\ s_n$ sont des nombres premiers différents, si l'on désigne, d'une manière générale par Ξ_{θ_h} le groupe formé par l'ensemble des éléments du

* Werke, Bd. II, p. 267.

groupe Ω qui appartiennent à des exposants égaux à des puissances de s_k, y compris la puissance s_k^θ, l'ordre du groupe Ξ_{θ_k} sera égal à $s_k^{\sigma_k}$ et l'on aura

$$\Omega = \Xi_{\theta_1}\Xi_{\theta_2}\ldots\Xi_{\theta_n}.$$

En particulier, si l'on a

$$\omega = s^\sigma$$

on aura

$$\Omega = \Xi_\theta.$$

Tout groupe eulérien dont l'ordre est une puissance de nombre premier est dit *uniprime*. Ce sont donc les groupes uniprimes que nous avons étudiés sous le nom de groupes Ξ_θ dans les paragraphes précédents.

23.

Nous avons vu que tout groupe monobase dont l'ordre est une puissance de nombre premier est simple. Je dis que tout groupe simple est monobase. En effet, l'ordre de tout groupe simple A est une puissance de nombre premier, car autrement le groupe A serait décomposable en un produit de plusieurs groupes uniprimes ayant pour ordres des nombres supérieurs à l'unité. Le groupe A est donc uniprime et par suite il est monobase ou égal à un produit de groupes monobases. La dernière supposition étant à excluse, il est clair que le groupe A sera monobase. Si donc on demande de décomposer un groupe Ω d'ordre p^π en groupes simples, le problème est identique à la décomposition du groupe Ω en groupes monobases. Il n'en est pas de même quand l'ordre du groupe Ω est divisible par deux nombres premiers différents. Soit, en effet,

$$\omega = p_1^{\pi_1}p_2^{\pi_2}\ldots p_n^{\pi_n}$$

l'ordre du groupe Ω. On aura, comme nous l'avons vu,

$$\Omega = \Xi_{\theta_1}\Xi_{\theta_2}\ldots\Xi_{\theta_n}$$

où le groupe Ξ_{θ_1} est d'ordre $p_1^{\pi_1}$; le groupe Ξ_{θ_2} d'ordre $p_2^{\pi_2}$, . . . le groupe Ξ_{θ_n} d'ordre $p_n^{\pi_n}$. Cela étant ainsi, décomposons les groupes Ξ en groupes simples et substituons ces décompositions dans l'expression

$$\Omega = \Xi_{\theta_1}\Xi_{\theta_2}\ldots\Xi_{\theta_n}$$

nous obtiendrons une décomposition de Ω en groupes simples. Toutes les décompositions de Ω en groupes simples peuvent d'ailleurs s'obtenir de cette manière.

En effet, si dans une telle décomposition on réunit en un seul tous les groupes dont les ordres sont des puissances d'un même nombre premier, on obtiendra une décomposition de Ω en groupes uniprimes dont les ordres seront premiers entre eux. Or une telle décomposition est complètement déterminée. *Donc, etc.* Une décomposition de Ω en groupes simples sera naturellement aussi une décomposition de Ω en groupes monobases. Soit maintenant A_1 un groupe simple faisant partie de la décomposition et ayant pour ordre une puissance de p_1, $p_1^{\sigma_1}$ par exemple. Il est clair qu'il y aura au moins un groupe simple de cette espèce provenant de la décomposition de Ξ_{θ_1}. Il y aura de même dans la décomposition de Ω un groupe simple A_2 d'un ordre $p_2^{\sigma_2}$. Cela étant ainsi, soit

$$A_1 A_2 = B \quad (\text{gr. } \Omega)$$

je dis que le groupe B sera monobase. En effet, soit a_1 une racine primitive du groupe A_1 et de même a_2 une racine primitive du groupe A_2, l'élément $a_1 a_2$ du groupe B appartiendra à l'exposant $p_1^{\sigma_1} p_2^{\sigma_2}$ et sera par conséquent une racine primitive du groupe B. En effectuant la combinaison des groupes A_1 et A_2 on diminue d'une unité le nombre des groupes monobases de la décomposition de Ω. En général, on peut réunir en un seul groupe monobase, deux ou plusieurs groupes monobases dont les ordres sont premiers entre eux. Toutes les décompositions de Ω en groupes monobases peuvent d'ailleurs s'obtenir de la manière indiquée. En effet, si dans une décomposition de Ω en groupes monobases, on décompose tous les groupes monobases en groupes uniprimes d'ordres premiers entre eux, on aura une décomposition de Ω en groupes simples. Désignons maintenant par

$$\begin{aligned}
\Xi_1 &= A_1' A_1'' \ldots \ldots A_1^{(u_1)}, \\
\Xi_2 &= A_2' A_2'' \ldots \ldots A_2^{(u_2)}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\Xi_n &= A_n' A_n'' \ldots \ldots A_n^{(u_n)}
\end{aligned}$$

les décompositions des groupes Ξ en groupes simples. On peut d'ailleurs supposer que les groupes Ξ sont rangés dans un ordre tel que

$$u_1 \geqq u_2 \geqq \ldots \geqq u_n.$$

Il est clair que le nombre de groupes monobases d'une décomposition de Ω en groupes monobases

$$\Omega = M_1 M_2 \ldots \ldots M_w$$

est au moins égal à u_1, car si l'on décompose tout groupe M_k de cette décomposition en groupes uniprimes et que l'on désigne par C_k le groupe uniprime dont les éléments font partie de Ξ_1, on aura

$$\Xi_1 = C_1 C_2 \ldots . C_w$$

d'où

$$w \geqq u_1.$$

On peut cependant décomposer le groupe Ω en u_1 groupes monobases

$$\Omega = B_1 B_2 \ldots . B_{u_1}.$$

En effet, pour cela il suffit de poser

$$\begin{aligned} B_1 &= A_1' A_2' \quad \ldots . A_n', \\ B_2 &= A_1'' A_2'' \quad \ldots . A_n'', \\ &\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots \\ B_{u_1} &= A_1^{(u_1)} A_2^{(u_1)} \ldots . A_n^{(u_1)} \end{aligned}$$

où il faut supprimer des expressions telles que $A_k^{(h)}$ où $h > u_k$. Le nombre de groupes monobases en lesquels il est possible de décomposer le groupe Ω varie donc de u_1 à $u_1 + u_2 + \ldots . u_n$, car en partant d'une décomposition de Ω en groupes simples—laquelle décomposition contiendra $u_1 + u_2 + \ldots . u_n$ groupes monobases—on peut arriver à la décomposition de Ω en u_1 groupes monobases par l'application successive du procédé consistant à réunir deux groupes monobases en un seul. Un groupe tel que Ω qu'il est possible de décomposer en u_1 groupes monobases et qu'il n'est pas possible de décomposer en un nombre moindre de groupes monobases, est dit *à u_1 bases.* D'une manière générale, tout groupe Ω pour lequel $u_1 > 1$ est dit *polybase.* La condition nécessaire et suffisante pour qu'un groupe Ω d'ordre

$$\omega = p_1^{\pi_1} p_2^{\pi_2} \ldots . p_n^{\pi_n}$$

où

$$p_1, p_2, \ldots . p_n$$

sont des nombres premiers différents, soit monobase est que ses groupes uniprimes

$$\Xi_1, \Xi_2, \ldots . \Xi_n$$

d'ordres

$$p_1^{\pi_1}, p_2^{\pi_2}, \ldots . p_n^{\pi_n}$$

soient tous monobases. Cette condition peut être remplacer par celle-ci. Pour

que le groupe Ω soit monobase, il faut et il suffit que l'égalité

$$x^{p_k} = 1 \quad (\text{gr. } \Omega)$$

ait p_k racines et pas plus quelle que soit la valeur de k depuis 1 jusqu'à n.

Parmi les manières de décomposer Ω en groupes monobases, et y en a une de particulièrement remarquable, car c'est la plus facile à obtenir théoriquement, si l'on veut faire abstraction du théorème fondamental de Gauss. En supposant, comme précédemment

$$u_1 \geqq u_2 > \ldots . \geqq u_n$$

et de plus que les ordres des groupes $A_k', A_k'', \ldots . A_k^{(u_k)}$ n'aillent pas en diminuant, on posera

$$\begin{aligned} B_1 &= A_1' A_2' \quad \ldots . A_n' \\ B_2 &= A_1'' A_2'' \quad \ldots . A_n'' \\ &\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots \\ B_{u_1} &= A_1^{(u_1)} A_2^{(u_1)} \ldots . A_n^{(u_1)} \end{aligned}$$

et
$$\Omega = B_1 B_2 \ldots . B_{u_1}.$$

On voit que l'ordre du groupe B_1 est divisible par celui de B_2, celui de B_2 par celui de B_3 et ainsi de suite. C'est la décomposition de M. Schering et de M. Kronecker.

En disant qu'un groupe Ω est à u_1 bases, on est loin d'en caractériser complètement la nature, mais cette dénomination se justifie par le rôle que joue le nombre u_1 dans les applications.

Soit donc, comme dans ce qui précède,

$$\Omega = B_1 B_2 \ldots . B_{u_1}$$

désignons par
$$b_1, b_2, \ldots . b_{u_1}$$

des racines primitives des groupes

$$B_1, B_2, \ldots . B_{u_1}$$

et par
$$h_1, h_2, \ldots . h_{u_1}$$

leurs ordres, l'expression

$$b_1^{v_1} b_2^{v_2} \ldots . b_{u_1}^{v_{u_1}}$$

où

$$\begin{array}{l} v_1 \text{ varie de } 0 \text{ à } h_1 - 1, \\ v_2 \quad \text{"} \quad \text{"} \; 0 \text{ à } h_2 - 1, \\ \dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots \\ v_{u_1} \quad \text{"} \quad \text{"} \; 0 \text{ à } h_{u_1} - 1 \end{array}$$

représentera tous les éléments du groupe Ω et chacun une seule fois.

Je dis maintenant que si une expression telle que

$$a_1^{v_1} a_2^{v_2} \dots a_m^{v_m}$$

est susceptible de représenter tous les éléments du groupe Ω, on a

$$m \geqq u_1.$$

En effet, tous les éléments du groupe Ξ_{θ_1} sont susceptibles d'être représentés par l'expression

$$a_1^{v_1} a_2^{v_2} \dots a_m^{v_m}.$$

Or si l'on pose

$$a_k = c_{k,1},\ c_{k,2}, \dots c_{k,n}$$

où $c_{k,l}$ est un élément du groupe Ξ_{θ_l},

$$c_l = c_{1,l}^{v_1} c_{2,l}^{v_2} \dots c_{m,l}^{v_m}$$

représentera un élément du groupe Ξ_{θ_l}. Tout élément a du groupe Ξ_{θ_1} étant susceptible d'être représenté par l'expression

$$a_1^{v_1} a_2^{v_2} \dots a_m^{v_m}$$

apparaîtra ainsi sous la forme

$$a = c_1 c_2 \dots c_n$$

où c_l est un élément du groupe Ξ_{θ_l}. On aura donc dans ce cas

$$c_2 = 1,\ c_3 = 1, \dots c_n = 1$$

et l'expression

$$a_1^{v_1} a_2^{v_2} \dots a_m^{v_m}$$

se réduira à celle-ci

$$c_{1,1}^{v_1},\ c_{2,1}^{v_2}, \dots c_{m,1}^{v_m}$$

qui sera ainsi susceptible de représenter tous les éléments du groupe Ξ_{θ_1}. D'où

$$m \geqq u_1.$$

C. Q. F. D.

24.

Je ne m'arrêterai pas à la déduction des théorèmes de Fermat, Wilson et d'autres propositions sur les groupes eulériens qu'on trouve dans les éléments d'arithmétique. Comme d'ailleurs le théorème fondamental de Gauss ramène la théorie des groupes eulériens quelconques à celle des groupes uniprimes, il est inutile de nous y attarder, et je passe donc rapidement aux applications spéciales que j'ai en vue.

25.

Si, en exécutant une opération univoque ϕ sur deux suites de nombres

$$\begin{matrix} a\,, b, \ldots . h \\ m, n, \ldots . z \end{matrix}$$

on obtient le même résultat, on peut écrire soit

$$\phi(a, b, \ldots . h) = \phi(m, n, \ldots . z)$$

soit $$a, b, \ldots . h = m, n, \ldots . z \qquad (\phi)$$

où la lettre ϕ entre parenthèses veut dire, que de chaque côté il faut exécuter l'opération ϕ. Dans ce second cas, il est d'usage d'écrire

$$a, b, \ldots . h \sim m, n, \ldots . z \qquad (\phi)$$

et de dire qu'on a remplacé une égalité par une équivalence. Désignons, par exemple, par $\phi(a, b)$ le quotient de $2^a 3^b$ par la plus haute puissance de b qui le divise, il est facile de prouver que l'équivalence

$$a, b \sim m, n \qquad (\phi)$$

entraîne l'égalité $$a - b = m - n$$

et inversement.* De même, si l'on désigne par $\phi_q(a)$, l'opération qui consiste à obtenir le reste de division du nombre a par un nombre constant q, on peut écrire $$a \sim b \qquad (\phi_q)$$

si en divisant tant a que b par q, on obtient le même reste. Gauss écrit†

$$a \equiv b \quad (\text{mod. } q)$$

*Cf. M. Kronecker: *Ueber den Zahlbegriff* (Philosophische Aufsätze an Zeller. Leipzig, 1887, p. 272).

† Disquisitiones arithmeticae, art. 2.

et il dit qu'on a une *congruence.** Les nombres a et b sont dits *congrus* l'un à l'autre ou entre eux suivant le module q. Les nombres a et b peuvent d'ailleurs être donnés implicitement, dans ce cas une congruence telle que

$$a \equiv b \pmod{q}$$

veut dire qu'après avoir exécuté toutes les opérations qu'il faut pour obtenir a et b, les resultats seront congrus entre eux suivant le module q.

En partant de la définition de la congruence, on établit sans peine les opérations qu'on peut exécuter sur des congruences. D'abord, la congruence

$$a \equiv b \pmod{q}$$

entraîne celle-ci $$b \equiv a \pmod{q}$$

et les deux congruences $$a \equiv c \pmod{q},$$
$$b \equiv c \pmod{q}$$

entraînent celle-ci $$a \equiv b \pmod{q}.$$

Si $$A \equiv a \pmod{q},\ B \equiv b \pmod{q}, \text{ etc.}$$

on aura $$A + B + \ldots . \equiv a + b + \ldots . \pmod{q}.$$

Si $$A \equiv a \pmod{q},\ B \equiv b \pmod{q}$$

on aura $$A - B \equiv a - b \pmod{q}.$$

Si $$A \equiv a \pmod{q},\ B \equiv b \pmod{q}$$

on aura $$AB \equiv \overline{ab} \pmod{q}$$

et de même, si l'on a

$$A \equiv a \pmod{q},\ B \equiv b \pmod{q},\ C \equiv c \pmod{q}, \text{ etc.}$$

on aura $$ABC \ldots . \equiv \overline{abc} \ldots . \pmod{q}.$$

Enfin, si $$A \equiv a \pmod{q}$$

et k est un entier positif, on aura

$$A^k \equiv a^k \pmod{q}.$$

Il résulte de là que si dans un polynôme tel que

$$Ax^a + Bx^b + Cx^c + \ldots .$$

on substitue à la place de x des nombres congrus entre eux, les résultats seront congrus entre eux.† Les nombres qui figurent dans les congruences peuvent

* C'est avec raison qu'un certain Professeur Hipp de Hambourg traduisait le mot *congruentia* par *Gleichrestigkeit* (Lettre de Schumacher à Gauss du 30 décembre 1809).

† Disquisitiones arithmeticae, art. 6, 7, 8, 9.

d'ailleurs être positifs, égaux à zéro ou négatifs sauf les exposans qui doivent toujours être positifs.

26.

Soit u un nombre entier positif quelconque, désignons par U l'ensemble de tous les nombres entiers positifs premiers à u et non supérieurs à u et par $\psi(u)$ leur nombre. Soient a et b deux nombres de l'ensemble U, le produit ab sera premier à u et par suite si l'on divise ab par u on obtient un reste c faisant partie de l'ensemble U. On peut écrire, en se servant de la notation de Gauss

$$ab \equiv c \quad (\text{mod. } u).$$

Nous considérerons tous les nombres de l'ensemble U comme des éléments d'un groupe U et nous dirons qu'en composant a et b on obtient c, ce qui peut s'écrire

$$ab = c \quad (\text{gr. } U).$$

L'expression ab ne veut plus dire ici qu'il faut multiplier a par b, mais qu'il faut composer a et b. En vertu des propriétés de la multiplication et des congruences, le groupe U sera un groupe eulérien d'ordre $\psi(u)$. Soit t un nombre premier impair quelconque, il est clair que l'étude de l'égalité

$$x^t = 1 \quad (\text{gr. } U)$$

peut être remplacée par celle de la congruence

$$x^t \equiv 1 \quad (\text{mod. } u)$$

où, selon Gauss, on ne considère que comme une seule et unique solution tous les nombres satisfaisant à la congruence qui ne diffèrent que par des multiples de u. En effet, parmi tous ces nombres il ne doit y avoir qu'un et un seul qui sera non supérieur à u,* lequel nombre sera aussi premier à u, en vertu de la congruence

$$x^t \equiv 1 \quad (\text{mod. } u).$$

Le nombre 1 joue d'ailleurs le rôle d'élément-unité dans le groupe U. Considérons donc la congruence

$$x^t \equiv 1 \quad (\text{mod. } u)$$

qu'on peut aussi mettre sous la forme

$$x^t - 1 = (x - 1)\, X(x) \equiv 1 \quad (\text{mod. } u).$$

* Disquisitiones arithmeticae, art. 26 et 4.

Si t ne divise pas $\psi(u)$, le groupe Ξ_θ formé par l'ensemble de tous les éléments du groupe U qui appartiennent à des exposants égaux à des puissances de t, se réduit au groupe-unité et la seule et unique solution de l'égalité

$$x^t = 1 \qquad (\text{gr. } U)$$

sera
$$x = 1.$$

Il s'ensuit que la seule et unique solution de la congruence

$$x^t \equiv 1 \qquad (\text{mod. } u)$$

sera
$$x \equiv 1 \qquad (\text{mod. } u).$$

Mais quand t divise $\psi(u)$, l'ordre du groupe Ξ_θ sera la plus haute puissance de t—désignons-la par t^r—qui divise $\psi(u)$. Le groupe U renfermera alors des éléments appartenant à des puissances de t autres que l'unité et en particulier à la puissance t. On pourra d'ailleurs remplacer l'égalité

$$x^t = 1 \qquad (\text{gr. } U)$$

par l'égalité
$$x^t = 1 \qquad (\text{gr. } \Xi_\theta)$$

où θ désigne naturellement le rang du groupe U par rapport au nombre premier t. Soit donc a une solution de l'égalité

$$x^t = 1 \qquad (\text{gr. } \Xi_\theta)$$

différente de l'élément-unité, on aura

$$a^t - 1 = (a-1)X(a) \equiv 0 \qquad (\text{mod. } u).$$

Le nombre $a-1$ ne pouvant pas être divisible par u, il y a deux cas à considérer:

1°. $X(a)$ est divisible par u.

2°. $X(a)$ n'est pas divisible par u, mais $a-1$ et $X(a)$ sont respectivement divisibles par deux diviseurs complémentaires h et k de u.

Arrêtons-nous au premier cas. On aura alors

$$X(a) \equiv 0 \qquad (\text{mod. } u).$$

Je pose maintenant

$$(x-a)(x-a^2)\ldots\ldots(x-a^{t-1}) = x^{t-1} - f_1 x^{t-2} + \ldots\ldots - f_{t-2} + f_{t-1}$$

et je cherche à évaluer les coefficients (mod. u). On aura d'abord

$$f_1 = a + a^2 + \ldots\ldots a^{t-1} = X(a) - 1 \equiv -1 \qquad (\text{mod. } u).$$

Cherchons le nombre des solutions de la congruence

$$x_1 + x_2 + \ldots . x_n \equiv h \quad (\text{mod. } t)$$

où h désigne un nombre donné pris dans la suite

$$0, 1, 2, \ldots . t-1$$

et

$$x_1, x_2, \ldots . x_n$$

sont des nombres cherchés qui doivent être différents entre eux et faire partie de la suite

$$0, 1, 2, \ldots . t-1.$$

Quant au nombre n, il est supposé satisfaisant aux conditions

$$0 < n < t-1.$$

Deux solutions $x_1, x_2, \ldots . x_n$ et $x'_1, x'_2, \ldots . x'_n$ ne sont d'ailleurs considérées comme distinctes que dans le cas où une des solutions ne peut être obtenue par la permutation des nombres de l'autre. Soit k un nombre de la suite

$$0, 1, 2, \ldots . t-1$$

différent de h, je considère de même une autre congruence

$$y_1 + y_2 + \ldots . y_n \equiv k \quad (\text{mod. } t)$$

analogue à la précédente. Je dis qu'à chaque solution de la congruence

$$x_1 + x_2 + \ldots . x_n \equiv h \quad (\text{mod. } t)$$

on peut faire correspondre une solution bien déterminée de la congruence

$$y_1 + y_2 + \ldots . y_n \equiv k \quad (\text{mod. } t)$$

et inversement. En effet, posons

$$\begin{aligned} nh_1 &\equiv h \quad (\text{mod. } t), \\ nk_1 &\equiv k \quad (\text{mod. } t) \end{aligned}$$

où h_1 et k_1 sont des nombres de la suite

$$0, 1, 2, \ldots . t-1.$$

Nous savons par la théorie des groupes eulériens complétée par la considération

des congruences où figurent des nombres non premiers au module que les nombres h_1 et k_1 seront complètement déterminés.* Cela étant ainsi, je pose

$$x_m \equiv y_m + h_1 - k_1 \quad (\text{mod. } t)$$

et par suite

$$y_m \equiv x_m + k_1 - h_1 \quad (\text{mod. } t)$$

le nombre m étant tout nombre de la suite

$$1, 2, \ldots . n.$$

De cette manière je fais correspondre à chaque solution de la congruence

$$x_1 + x_2 + \ldots . x_n \equiv h \quad (\text{mod. } t)$$

une solution bien déterminée de la congruence

$$y_1 + y_2 + \ldots . y_n \equiv k \quad (\text{mod. } t)$$

et inversement.† Les deux congruences auront donc le même nombre de solutions. Ce nombre est d'ailleurs facile à évaluer. En effet, le nombre des expressions telles que

$$x_1 + x_2 + \ldots . x_n$$

différant entre elles au moins par un des nombres qui figurent dans la somme, étant égal à

$$\frac{t!}{n!\,(t-n)!}$$

on obtiendra le nombre cherché en divisant $\frac{t!}{n!\,(t-n)!}$ par t, ce qui donne

$$\frac{(t-1)!}{n!\,(t-n)!}.$$

J'aborde maintenant une congruence telle que

$$x_1 + x_2 + \ldots . x_n \equiv h \quad (\text{mod. } t)$$

où

$$x_1, x_2, \ldots . x_n$$

sont des nombres de la suite

$$1, 2, \ldots . t-1$$

et je désigne par $\chi_h^{(n)}$ le nombre de ses solutions. Quand $n > 1$, toute solution de la congruence

$$x_1 + x_2 + \ldots . x_n \equiv h \quad (\text{mod. } t)$$

où

$$x_1, x_2, \ldots . x_n$$

* Disquisitiones arithmeticae, art. 26. † Ibid. art. 8.

font partie de la suite

$$0,\ 1,\ 2,\ \ldots.\ t-1$$

est *soit* une solution de la congruence

$$x_1+x_2+\ldots.x_n\equiv h \quad (\text{mod. } t)$$

où les indéterminées ne peuvent pas prendre la valeur 0, *soit* par la suppression de l'indéterminée qui est égale à zéro, une solution de la congruence

$$x_1+x_2+\ldots.x_{n-1}\equiv h \quad (\text{mod. } t)$$

où les indéterminées ont encore des valeurs différentes de zéro. L'inverse ayant aussi lieu, on aura

$$\chi_h^{(n-1)}+\chi_h^{(n)}=\frac{(t-1)!}{n!\,(t-n)!}$$

pour $n>1$.

Or on obtient immédiatement

$$\chi_0^{(1)}=0.$$

et

$$\chi_h^{(1)}=1$$

pour $h>0$; on aura donc

$$\chi_0^{(n)}=\frac{(t-1)!}{n!\,(t-n)!}-\frac{(t-1)!}{(n-1)!\,(t-n+1)!}+\ldots.+(-1)^{n-2}\frac{(t-1)!}{2!\,(t-2)!}$$

et*

$$\chi_h^{(n)}=\frac{(t-1)!}{n!\,(t-n)!}-\frac{(t-1)!}{(n-1)!\,(t-n+1)!}+\ldots.$$
$$+(-1)^{n-2}\frac{(t-1)!}{2!\,(t-2)!}+(-1)^{n-1}$$

d'où

$$\chi_h^{(n)}=\chi_0^{(n)}-(-1)^n.$$

Cela étant ainsi, on aura

$$f_n\equiv\chi_0^{(n)}+(a+a^2+\ldots.a^{t-1})\,\chi_h^{(n)}\equiv(-1)^n+X(a)\,\chi_h^{(n)}\equiv(-1)^n \quad (\text{mod. } u).$$

Comme on a d'ailleurs

$$f_{t-1}=\prod_{h=1}^{h=\frac{t-1}{2}}a^h a^{t-h}\equiv 1 \quad (\text{mod. } u)$$

on aura $\quad (x-a)(x-a^2)\ldots.(x-a^{t-1})\equiv X(x) \quad (\text{mod. } u)$

---

*La sommation donne

$$\chi_h^{(n)}=\frac{\dfrac{(t-1)!}{n!\,(t-n-1)!}-(-1)^n}{t}.$$

dans ce sens que tout coëfficient de $X(x)$ sera congru au coefficient correspondant dans le développement de $(x-a)(x-a^2)\ldots.(x-a^{t-1})$.

Montrons maintenant qu'il existe des modules u pour lesquels la congruence

$$X(x)\equiv 0 \pmod{u}$$

est possible. En premier lieu, si u est égal à un nombre premier q tel que t divise $\psi(q)$, l'égalité

$$x^t = 1 \quad (\text{gr. } Q)$$

où Q désigne le groupe formé par l'ensemble de tous les nombres positifs premiers à q et non supérieurs à q, admet nécessairement une solution a différente de l'élément-unité. L'expression

$$a^t - 1 = (a-1)X(a)$$

sera alors divisible par q. Or nous savons que la divisibilité d'un produit par un nombre premier suppose la divisibilité d'au moins un des facteurs par ce nombre premier.* La solution a étant différente de 1, la différence $a-1$ ne peut être divisible par q; c'est donc le facteur $X(a)$ qui doit l'être et l'on aura

$$X(a)\equiv 0 \pmod{q}.$$

De même, si le module u est égal à une puissance q^w d'un nombre premier q pour lequel t divise $\psi(q)$, la congruence

$$X(x)\equiv 0 \pmod{q^w}$$

admet nécessairement une solution. En effet, soit b une solution de la congruence

$$x^t\equiv 1 \pmod{q}$$

différente de l'unité, on aura

$$X(b)\equiv 0 \pmod{q}.$$

Désignons par $q^\varkappa$ la plus haute puissance de q qui divise $X(b)$, si $\varkappa \geqq w$, on peut faire $a=b$, si non posons

$$b^t\equiv 1+hq^\varkappa \pmod{q^{\varkappa+1}}$$

ce qu'on peut toujours faire en vertu de ce que $b^t-1=(b-1)X(b)$ est divisible par $q^\varkappa$. Je dis que h n'est pas divisible par q, car dans le cas contraire $b^t-1=(b-1)X(b)$ serait divisible par $q^{\varkappa+1}$ et comme $b-1$ n'est pas divisible

* Euclidis Elementa, ed. Heiberg VII, 30, t. II, p. 248.

par q, $X(b)$ serait divisible par $q^{\kappa+1}$ contrairement à la supposition. On aura maintenant pour toute valeur de k

$$(b+kq^\kappa)^t \equiv 1 + hq^\kappa + tb^{t-1}kq^\kappa \quad (\text{mod. } q^{\kappa+1}).$$

Je pose maintenant

$$h + ktb^{t-1} \equiv 0 \quad (\text{mod. } q)$$

et je détermine k par cette congruence ce qui est toujours possible car t ne peut avoir un facteur commun avec q que dans le cas où $t=q$; or dans ce cas t ne pourrait diviser $\psi(q)$, les nombres premiers à q et non supérieurs à q devant être cherchés dans la suite

$$1, 2, \ldots . q-1.$$

Le nombre k étant ainsi choisi, on aura

$$(b+kq^\kappa)^t \equiv 1 \quad (\text{mod. } q^{\kappa+1})$$

et l'expression

$$(b+kq^\kappa)^t - 1 = (b+kq^\kappa - 1)\, X(b+kq^\kappa)$$

sera ainsi divisible par $q^{\kappa+1}$. Comme $b+kq^\kappa-1$ n'est pas divisible par q, on aura

$$X(b+kq^\kappa) \equiv 0 \quad (\text{mod. } q^{\kappa+1}).$$

Le nombre k étant premier à q et si l'on veut non supérieur à q, l'expression $b+kq^\kappa$ sera première à $q^{\kappa+1}$ et non supérieure à $q^{\kappa+1}$. En procédant ainsi de proche en proche, on trouvera une solution a de la congruence

$$X(x) \equiv 0 \quad (\text{mod. } q^{\kappa+1})$$

première à $q^{\kappa+1}$ et non supérieure à $q^{\kappa+1}$.

D'une manière générale, si l'on a

$$u = q_1^{\varpi_1} q_2^{\varpi_2} \ldots . q_\nu^{\varpi_\nu}$$

où t divise tous les nombres

$$\psi(q_1), \psi(q_2), \ldots . \psi(q_\nu)$$

les nombres $q_1, q_2, \ldots . q_\nu$ étant d'ailleurs des nombres premiers différents entre eux, on peut trouver une solution a de la congruence

$$X(x) \equiv 0 \quad (\text{mod. } u)$$

et par suite effectuer la décomposition

$$X(x) \equiv (x-a)(x-a^2) \ldots . (x-a^{t-1}) \quad (\text{mod. } u).$$

En effet, si a_1 est une solution de l'égalité

$$X(x) \equiv 0 \quad (\text{mod. } q_1^{w_1})$$

a_2 une solution de l'égalité

$$X(x) \equiv 0 \quad (\text{mod. } q_2^{w_2}),$$

$$\dots\dots\dots\dots$$

enfin a_v une solution de l'égalité

$$X(x) \equiv 0 \quad (\text{mod. } q_v^{w_v})$$

on n'aura qu'à faire*

$$\begin{aligned} a &\equiv a_1 \quad (\text{mod. } q_1^{w_1}), \\ a &\equiv a_2 \quad (\text{mod. } q_2^{w_2}), \\ &\dots\dots\dots\dots \\ a &\equiv a_v \quad (\text{mod. } q_v^{w_v}) \end{aligned}$$

et l'on aura $\quad X(a) \equiv 0 \quad (\text{mod. } u)$

et par suite

$$X(x) \equiv (x-a)(x-a^2) \dots (x-a^{t-1}) \quad (\text{mod. } u).$$

Il est facile d'évaluer le nombre des solutions de la congruence

$$x^t \equiv 1 \quad (\text{mod. } u)$$

pour tout module u de la forme $q_1^{w_1} q_2^{w_2} \dots q_v^{w_v}$ où $q_1, q_2, \dots q_v$ sont des nombres premiers différents tels que t divise tous les $\psi(q)$. En effet, pour $u = q$ on aura $\quad x^t - 1 \equiv (x-1)(x-a)(x-a^2) \dots (x-a^{t-1}) \quad (\text{mod. } q).$

Or un produit ne peut être divisible par un nombre premier à moins qu'un des facteurs ne le soit; on aura donc les solutions

$$\begin{aligned} x &\equiv 1 \quad (\text{mod. } q), \\ x &\equiv a \quad (\text{mod. } q), \\ x &\equiv a^2 \quad (\text{mod. } q), \\ &\dots\dots\dots\dots \\ x &\equiv a^{t-1} \quad (\text{mod. } q). \end{aligned}$$

Ces solutions sont d'ailleurs toutes différentes entre elles, car si l'on avait

$$a^h \equiv a^k \quad (\text{mod. } q),$$

par exemple, où h et k sont des nombres différents pris dans la suite

$$0, 1, 2, \dots t-1$$

---

* Disquisitiones arithmeticae, art. 36.

et $h > k$; on en tirerait

$$a^{h-k} \equiv 1 \pmod{q}$$

contrairement à la supposition que a appartient à l'exposant t. Quand on a

$$u = q^w$$

on a encore

$$x^t - 1 = (x-1)(x-a)(x-a^2)\ldots\ldots(x-a^{t-1}) \pmod{q^w}$$

et comme deux facteurs de ce produit ne peuvent pas être divisible par q en même temps, comme nous venons de le voir, il faut qu'un des facteurs soit divisible par q^w. On en tire les seules et uniques solutions

$$\begin{array}{ll} x \equiv 1 & \pmod{q^w}, \\ x \equiv a & \pmod{q^w}, \\ x \equiv a^2 & \pmod{q^w}, \\ \ldots\ldots & \\ x \equiv a^{t-1} & \pmod{q^w} \end{array}$$

qui seront encore différentes entre elles. Le nombre des solutions de la congruence

$$x^t \equiv 1 \pmod{q^w}$$

est donc égal à t. Enfin pour

$$u = q_1^{w_1} q_2^{w_2} \ldots\ldots q_v^{w_v}$$

ou aura de même

$$x^t - 1 \equiv (x-1)(x-a)(x-a^2)\ldots\ldots(x-a^{t-1}) \pmod{u}.$$

Or comme deux facteurs du produit précédent ne peuvent être divisible en même temps par un même nombre premier q, il s'agit de distribuer toutes les puissances de nombre premier q^w parmi les facteurs du produit

$$(x-1)(x-a)(x-a^2)\ldots\ldots(x-a^{t-1})$$

avec la faculté d'attribuer plusieurs puissances de nombre premier à un seul facteur. Comme il y a ainsi t alternatives pour chaque puissance de nombre premier, cela nous donne en tout

$$t^v$$

solutions, qui seront d'ailleurs toutes différentes entre elles, en vertu de l'article 36 des *Disquisitiones arithmeticae*.

Il existe des modules pour lesquels la décomposition de $X(x)$ en facteurs linéaires est impossible. En effet, posons d'abord

$$u = t$$

et désignons par T le groupe formé par l'ensemble des nombres premiers à t et non supérieurs à t, l'égalité

$$x^t = 1 \quad (\text{gr. } T)$$

n'admettra qu'une seule et unique solution

$$x = 1 \quad (\text{gr. } T)$$

car $\psi(t)$ n'est pas divisible par t. De même la congruence

$$x^t \equiv 1 \quad (\text{mod. } t)$$

n'aura qu'une seule et unique solution

$$x \equiv 1 \quad (\text{mod. } t).$$

Or comme on a $\quad X(1) = t \equiv 0 \quad (\text{mod. } t)$
on aura la décomposition

$$X(x) \equiv (x-1)^{t-1} \quad (\text{mod. } t)$$

et par suite $\quad x^t - 1 \equiv (x-1)^t \quad (\text{mod. } t).$
Mais quand $\quad u = t^\tau$
où $\tau > 1$, la congruence

$$X(x) \equiv 0 \quad (\text{mod. } t^\tau)$$

n'admet pas de solution et par suite n'est pas décomposable en un produit de facteurs linéaires. En effet, si l'on avait

$$X(a) \equiv 0 \quad (\text{mod. } t^\tau)$$

on en tirerait

$$X(x) \equiv (x-a)(x-a^2) \ldots . (x-a^{t-1}) \quad (\text{mod. } t^\tau)$$

et par suite

$$X(1) \equiv (1-a)(1-a^2) \ldots . (1-a^{t-1}) \quad (\text{mod. } t^\tau).$$

On devrait donc avoir, pour une certaine valeur de h comprise entre 1 et $t-1$ inclusivement $\quad a^h \equiv 1 \quad (\text{mod. } t)$

et par suite $\quad a^{mh+nt} \equiv 1 \quad (\text{mod. } t)$

où m et n sont des nombres entiers quelconques. On en tirerait

$$a^k \equiv 1 \pmod{t}$$

pour toute valeur de k prise dans la suite

$$1, 2, \ldots . t-1$$

et par suite $$t \equiv t^{\tau-1} \pmod{t^\tau}$$

congruence absurde puisque tant $t-1$ que τ sont supérieurs à l'unité. La congruence $$x^t \equiv 1 \pmod{t^\tau}$$

admet t solutions et pas plus car $\psi(t^\tau)$ est divisible par t et pas par une puissance supérieure de t. Or comme on a

$$x^t - 1 = (x-1)X(x)$$

on voit facilement que ces solutions sont

$$1, 1+t^{\tau-1}, 1+2t^{\tau-1}, \ldots, 1+(t-1)t^{\tau-1}$$

car toutes ces valeurs rendent $x-1$ divisible par $t^{\tau-1}$ et $X(x)$ divisible par t. Toutes ces solutions étant différentes entre elles, il ne pourra y en avoir d'autres. Il est clair que si l'on pose

$$a = 1 + kt^{\tau-1}$$

où k est pris dans la suite

$$1, 2, \ldots . t-1$$

toutes les solutions de la congruence

$$x^t \equiv 1 \pmod{t^\tau}$$

seront $$1, a, a^2, \ldots . a^{t-1}.$$

Mais on ne peut plus parler d'une décomposition telle que

$$x^t - 1 \equiv (x-1)(x-a)(x-a^2) \ldots . (x-a^{t-1}) \pmod{t^\tau}$$

car $$1 + a + a^2 + \ldots . a^{t-1}$$

est bien divisible par t mais non par t^2.

La congruence $$X(x) \equiv 0 \pmod{u}$$

n'est pas résoluble non plus quand le module est égal à un nombre premier r différent de t et tel que t ne divise pas $\psi(r)$. En effet, si l'on désigne par R le

groupe formé par l'ensemble des nombres premiers à r et non supérieurs à r, l'égalité

$$x^t = 1 \quad (\text{gr. } R)$$

n'admettra qu'une seule solution

$$x = 1.$$

De même la congruence

$$x^t \equiv 1 \quad (\text{mod. } r)$$

n'admettra qu'une seule et unique solution

$$x \equiv 1 \quad (\text{mod. } r).$$

Or comme on a $\qquad X(1) = t$

la congruence $\qquad X(x) \equiv 0 \quad (\text{mod. } r)$

n'admet pas la solution

$$x \equiv 1 \quad (\text{mod. } r)$$

et par suite elle n'en admettra aucune puisque toute solution de la congruence

$$X(x) \equiv 0 \quad (\text{mod. } r)$$

est nécessairement une solution de la congruence

$$x^t - 1 \equiv 0 \quad (\text{mod. } r).$$

On ne peut parler dans ce cas non plus d'une décomposition telle que

$$x^t - 1 \equiv (x-1)^t \quad (\text{mod. } r)$$

car le coefficient de x^{t-1} dans le développement de $(x-1)^t$ est égal à $-t$ et par suite non divisible par r. D'une manière générale, la congruence

$$X(x) \equiv 0 \quad (\text{mod. } u)$$

n'admet pas de solution et est indécomposable en facteurs linéaires pour tout module u divisible par un nombre premier r différent de t et tel que $\psi(r)$ n'est pas divisible par t. En somme, la congruence

$$X(x) \equiv 0 \quad (\text{mod. } u)$$

est résoluble et par suite décomposable en un produit de facteurs linéaires pour tout module u qui n'est divisible ni par t^2 ni par aucun nombre premier r différent de t et tel que t ne divise pas $\psi(r)$. En effet, un tel module u est soit de la forme

$$u = q_1^{w_1} q_2^{w_2} \ldots . q_v^{w_v}$$

où t divise tous les $\psi(q)$, soit de la forme

$$u = t q_1^{w_1} q_2^{w_2} \ldots . q_v^{w_v}$$

où t divise encore tous les $\psi(q)$. Dans le premier cas on n'a qu'à faire

$$a \equiv a_k \quad (\text{mod. } q_k^{\varpi_k}) \qquad (k = 1, 2, \ldots . v)$$

et dans le second

$$a \equiv 1 \quad (\text{mod. } t),$$
$$a \equiv a_k \quad (\text{mod. } q_k^{\varpi_k}) \qquad (k = 1, 2, \ldots . v)$$

et l'on aura

$$X(x) \equiv (x-a)(x-a^2) \ldots . (x-a^{t-1}) \qquad (\text{mod. } u).$$

Tout nombre M qui n'est divisible ni par t^2 ni par un nombre premier r différent de t et tel que t ne divise pas $\psi(r)$, peut donc être appelé *diviseur de* $X(x)$.

Cela étant ainsi, désignons par Ξ_1 le groupe formé par l'ensemble de toutes les solutions de l'égalité

$$x^t = 1 \quad (\text{gr. } U)$$

et par t^{m_1} son ordre, il est facile d'évaluer m_1. En effet, posons

$$u = t^\tau MN$$

où t^τ est la plus haute puissance de t qui divise u, le nombre τ pouvant d'ailleurs avoir la valeur 0, et M le produit de tous les diviseurs premiers q de $\frac{u}{t^\tau}$ tels que t divise $\psi(q)$, chaque nombre premier q devant d'ailleurs figurer dans M autant de fois que dans u. Le nombre N n'est donc autre chose que le quotient $\frac{u}{t^\tau M}$ qui peut se réduire éventuellement à un. Si l'on désigne par v le nombre de nombres premiers différents qui divisent M, la congruence

$$x^t \equiv 1 \quad (\text{mod. } M)$$

admettra, comme nous l'avons vu, t^v solutions. La congruence

$$x^t \equiv 1 \quad (\text{mod. } t^\tau)$$

admettra une seule solution dans le cas où $\tau < 2$ et en admettra t dans le cas où $\tau > 1$. Enfin si $N > 1$, la congruence

$$x^t \equiv 1 \quad (\text{mod. } N)$$

n'admet qu'une seule et unique solution, car t ne divise pas $\psi(N)$. Cela étant ainsi, en vertu de l'article 36 des *Disquisitiones arithmeticae* on aura

$$m_1 = v + \varepsilon$$

où ε est égal à 1 dans le cas où $\tau > 1$, il est égal à 0 dans tous les autres cas. Le groupe Ξ_1 est donc à $m_1 = v + \varepsilon$ bases où le nombre m_1 ne peut jamais surpasser le nombre de nombres premiers différents divisant u. Disons plus, comme on a

$$\psi(2^z) = 2^{z-1}$$

pour toute valeur positive et entière de z, le nombre m_1 ne peut jamais surpasser le nombre de nombres premiers impairs divisant u. Nous avons supposé le nombre u décomposé en un produit de la forme

$$t^\tau MN,$$

mais la théorie elle-même nous fournit cette décomposition. En effet, il résulte de la théorie précédente que le moindre commun multiple de tous les plus grands communs diviseurs de

$$X(x) \text{ et } u,$$

où x doit parcourir toutes les solutions de

$$x^t \equiv 1 \pmod{u},$$

est égal à tM ou à M suivant que τ est plus grand ou égal à 0. D'une manière analogue, la théorie elle-même nous fournit la décomposition de M en un produit de puissances de nombre premier premières entre elles. Il y a une différence essentielle entre le postulat III d'Euclide,* par exemple, et la supposition qu'on fait dès le début de la théorie des formes quadratiques qu'il est possible d'effectuer la décomposition de tout nombre donné en ses facteurs premiers. Le postulat III ne reçoit aucun perfectionnement à mesure qu'on avance dans la géométrie. Il n'en est pas de même de la décomposition d'un nombre en ses facteurs premiers. D'abord ce n'est qu'un tâtonnement fort rude, mais il le devient beaucoup moins à mesure qu'on avance dans la théorie des formes quadratiques. Il est donc de toute nécessité d'y revenir, une fois la théorie des formes quadratiques achevée. La sixième section des *Disquisitiones arithmeticae* n'est pas un hors-d'œuvre, c'est un complément absolument indispensable de toute théorie des formes quadratiques. Que l'on ramène la question qui consiste à déterminer le caractère quadratique d'un nombre donné a par rapport à un nombre donné m à la décomposition de m en ses facteurs premiers, je le veux bien. Mais cela fait,

* Elementa, ed. Heiberg, t. I, p. 8.

que l'on complète la théorie en montrant comment la décomposition de m en ses facteurs premiers dépend de la résolution de la congruence

$$x^2 \equiv a \pmod{m}.$$

Abordons maintenant la congruence*

$$x^2 \equiv 1 \pmod{u}.$$

Ici, il se présente quelque chose de particulier. En effet, l'expression $x^2 - 1$ est décomposable en facteurs linéaires algebriquement, car on a

$$x^2 - 1 = (x-1)(x+1).$$

On aura donc

$$x^2 - 1 \equiv (x-1)(x-u+1) \pmod{u}$$

où $u-1$ est, comme il est facile de voir, premier à u. S'il s'agit de rendre $x^2 - 1$ divisible par u, il suffit donc de rendre $x-1$ divisible par un diviseur h de u et $x-u+1$ divisible par un diviseur k de u tels qu'on ait

$$hk \equiv 0 \pmod{u}.$$

Je dis que h et k ne peuvent avoir d'autre facteur commun que 2. En effet, cela résulte immédiatement de la relation

$$(x-1) - (x-u+1) + u = 2.$$

Si donc le module u est impair, les nombres h et k ne peuvent avoir de facteur commun. Pour tout module u supérieur à 2, la congruence a d'ailleurs deux solutions bien distinctes

$$x \equiv 1 \pmod{u}$$

et

$$x \equiv u-1 \pmod{u}.$$

D'où il résulte que 2 divise toujours $\psi.(u)$ quelque soit le module u supérieur à 2. Si le module u est égal à une puissance de nombre premier impair q^w, la congruence

$$x^2 \equiv 1 \pmod{q^w}$$

n'a que deux solutions. En effet, le plus grand commun diviseur de $x-1$, $x-q^w+1$ et q étant égal 1 quelle que soit la valeur de x, il faut qu'un des deux facteurs $x-1$ et $x-q^w+1$ soit divisible par q^w pour que x^2-1 le soit, ce qui donne les deux seules et uniques solutions

$$x \equiv 1 \pmod{q^w},$$
$$x \equiv q^w - 1 \pmod{q^w}.$$

* Cette congruence a déjà été considérée par M. Schering dans son mémoire intitulé *Zur Theorie der quadratischen Reste* (Acta Mathematica, Vol. I, p. 153).

Quand on a $$u = q_1^{w_1} q_2^{w_2} \ldots q_v^{w_v}$$
où $$q_1^{w_1},\ q_2^{w_2},\ \ldots\ q_v^{w_v}$$
sont des puissances de nombre premier impair premières entre elles, on établira comme dans le cas de la congruence
$$x^t \equiv 1 \pmod{u}$$
que la congruence $$x^2 \equiv 1 \pmod{u}$$
admet 2^v solutions.

Considérons maintenant la congruence
$$x^2 \equiv 1 \pmod{2^z}.$$
Si $z = 1$, elle n'admet évidemment qu'une seule et unique solution
$$x \equiv 1 \pmod{2}.$$
Si $z = 2$, on a les solutions
$$x \equiv 1 \pmod{4},$$
$$x \equiv 3 \pmod{4}.$$
Comme ces solutions épuisent les nombres premiers à 4 et non supérieurs à 4, il ne pourra y en avoir d'autres. Nous pouvons donc nous borner à la considération du cas où $z > 2$. On aura donc la congruence
$$x^2 - 1 \equiv (x-1)(x - 2^z + 1) \equiv 0 \pmod{2^z}.$$
Comme on a $$(x - 2^z + 1) - (x - 1) \equiv 2 \pmod{2^z}$$
pour toute valeur de x, il est clair qu'un des deux nombres $x - 2^z + 1$ et $x - 1$ sera pairement pair et l'autre impairement pair pour toute valeur de x qui satisfait à la congruence
$$x^2 \equiv 1 \pmod{2^z}.$$
Le nombre pairement pair devra donc être divisible par 2^{z-1} et il sera par conséquent $$\equiv 0 \text{ ou } \equiv 2^{z-1} \pmod{2^z}.$$
Cela nous donne en tout quatre solutions
$$x \equiv 1 \pmod{2^z},$$
$$x \equiv 1 + 2^{z-1} \pmod{2^z},$$
$$x \equiv 2^z - 1 \pmod{2^z},$$
$$x \equiv 2^{z-1} - 1 \pmod{2^z}.$$
Toutes ces solutions sont distinctes entre elles, comme on peut s'en assurer en évaluant (mod. 2^z) leurs différences deux à deux. En vertu de l'article 36 des

Disquisitiones arithmeticae que nous avons déjà cité tant de fois, on aura donc la proposition suivante. La congruence

$$x^2 \equiv 1 \quad (\text{mod. } u)$$

admet 2^v solutions, si u est un nombre impair ou impairement pair divisible par v nombres premiers impairs différents. Le nombre de solutions est égal à 2^{v+1} si le nombre u est divisible par 4 sans l'être par 8, enfin le nombre de solutions est égal à 2^{v+2} quand le module u est divisible par 8.

La différence essentielle entre la congruence

$$x^2 \equiv 1 \quad (\text{mod. } u)$$

et la congruence

$$x^t \equiv 1 \quad (\text{mod. } u)$$

où t est un nombre premier impair, consiste donc en ce que $x^t - 1$ n'est pas décomposable en facteurs linéaires pour $u = t^\tau$ où $\tau > 1$, tandis que $x^2 - 1$ est algébriquement décomposable en facteurs linéaires et par suite aussi pour le module 2^s.

En résumé, si l'on désigne par Ξ_1 le groupe formé par l'ensemble des solutions de la congruence

$$x^s \equiv 1 \quad (\text{mod. } u)$$

où s est un nombre premier quelconque et par s^{m_1} son ordre, on aura, si

$$u = s^\sigma q_1^{w_1} q_2^{w_2} \ldots\ldots q_v^{w_v}$$

est la décomposition de u en puissances de nombre premier premières entre elles (la puissance s^σ pouvant se réduire éventuellement à l'unité),

$$m_1 = v$$

pour $s = 2$ $\sigma < 2$,

$$m_1 = v + 1$$

pour $s = 2$ $\sigma = 2$,

$$m_1 = v + 2$$

pour $s = 2$ $\sigma > 2$,

$$m_1 = v_t$$

pour $s = t > 2$, où v_t désigne le nombre des nombres de la suite

$$\psi(s^\sigma),\ \psi(q_1^{w_1}),\ \psi(q_2^{w_2}) \ldots . \psi(q_v^{w_v})$$

qui sont divisibles par s de sorte qu'on aura toujours

$$v_t \leqq v'$$

si l'on désigne par v' le nombre de nombres premiers impairs qui divisent u. Le groupe U sera par conséquent à autant de bases que le groupe formé par l'ensemble des solutions de la congruence

$$x^2 \equiv 1 \pmod{u}.$$

27.

Passons maintenant au théorème d'Euclide.* Si les nombres premiers étaient en nombre fini et se réduisaient par exemple aux nombres premiers

$$q_1, q_2, \dots q_n$$

qu'on peut supposer rangés par ordre de grandeur, le nombre $q_1 q_2 \dots q_n + 1$ serait ou premier ou divisible par un nombre premier $q_{n+1} > q_n$. Il résulte du raisonnement d'Euclide que, entre q_n exclusivement et $q_1 q_2 \dots q_n + 1$ inclusivement il existe au moins un nombre premier.

Voyons maintenant si, en se fondant sur les préliminaires établis dans les paragraphes précédents, il est possible de conclure quelque chose de plus.

Posons $$M = q_1 q_2 \dots q_n$$

le groupe formé par tous les nombres premiers à M et non supérieurs à M sera à $n-1$ bases. Or si l'on désigne par

$$q_{n+1}, q_{n+2}, \dots q_{n+h}$$

tous les nombres premiers compris entre q_n exclusivement et M, tous les nombres qui font partie du groupe $\mathfrak{M}$ pourront être représentés par la formule

$$q_{n+1}^{u_{n+1}} q_{n+2}^{u_{n+2}} \dots q_{n+h}^{u_{n+h}}$$

où les nombres u ont la valeur 0 ou une valeur entière positive. Il s'ensuit qu'on doit avoir, comme nous l'avons démontré dans les paragraphes précédents,

$$h \geqq n - 1.$$

Entre q_n exclusivement et M, il existe donc au moins $n-1$ nombres premiers. Si, dans le raisonnement d'Euclide, on remplace $M+1$ par $M-1$, il en résultera qu'entre q_n exclusivement et M, il existe au moins un seul nombre premier. La démonstration de M. Kummer (Monatsberichte der Berliner Academie) repose sur des considérations analogues à celles que nous venons de développer. M. Kummer

* Elem. IX, 20, ed. Heiberg, Vol. II, p. 388.

démontre d'abord que l'ensemble $\mathfrak{M}$ renferme au moins un élément et que par suite il faut au moins un nombre premier différent de

$$q_1, q_2, \ldots. q_n$$

pour qu'on puisse l'exprimer comme un produit de facteurs premiers. La démonstration du théorème d'Euclide que j'ai donnée il y a huit ans dans le *Bulletin de M. Darboux* repose sur d'autres principes. Il est facile de faire voir que dans la suite

$$1, 2, 3, \ldots. N$$

il y a plus de $\frac{1}{8}N$ nombres sans facteur quadratique. Si donc on désigne par

$$q_1, q_2, \ldots. q_m$$

les nombres premiers de la suite précédente, tout nombre sans facteur quadratique de la suite sera de la forme

$$q_1^{u_1} q_2^{u_2} \ldots. q_m^{u_m}$$

où tout exposant u_k a une des deux valeurs 0 et 1. Comme cette fois-ci les exposants des nombres premiers ont une limite supérieure, il n'est plus nécessaire de recourir à la théorie des groupes eulériens et l'on obtient immédiatement

$$m > \log. \text{acoust.} \tfrac{1}{8} N.$$

28.

Le raisonnement d'Euclide peut servir à démontrer l'existence de nombres premiers ayant certaines formes particulières données d'avance. Nous avons vu que t étant un nombre premier impair quelconque, l'expression

$$\frac{x^t - 1}{x - 1} = X(x) = x^{t-1} + x^{t-2} + \ldots. + x + 1$$

ne peut admettre comme diviseurs premiers que le nombre premier t et cela une seule fois, et des nombres premiers de la forme

$$ht + 1$$

où h est un nombre entier positif quelconque. En particulier, $X(t)$ n'étant pas divisible par t et étant supérieur à l'unité, sera nécessairement divisible par un nombre premier q_1 de la forme $ht + 1$. De même $X(tq_1)$ étant supérieur à 1 et n'étant divisible ni par t ni par q_1 sera nécessairement divisible par un nombre premier q_2 de la forme $ht + 1$ et différent de q_1. D'une manière analogue le

nombre $X(tq_1q_2)$ étant supérieur à 1 et n'étant divisible ni par t, ni par q_1 ni par q_2 sera divisible par un nombre premier q_3 de la forme $ht+1$ et différent tant de q_1 que de q_2.

En continuant de la même manière on peut trouver autant de nombres premiers de la forme $ht+1$

$$q_1, q_2, \ldots . q_n$$

que l'on veut. Il est donc bien démontré que le nombre de nombres premiers de la forme $ht+1$ peut surpasser tout nombre entier donné d'avance.* Nous avons supposé t impair; si l'on met dans la formule $ht+1$ à la place de t le seul et unique nombre premier pair 2, on obtient la formule

$$2h+1$$

qui convient à tout nombre premier impair. La proposition s'applique donc aussi à ce cas. Il convient maintenant de dire quelques mots sur les classes de nombres premiers de M. Lipschitz.† Le nombre premier 2 forme à lui seul la classe 1, à la classe 2 appartiennent tous les nombres premiers de la forme 2^n+1 et d'une manière générale on considère comme appartenant à la classe μ tout nombre premier q_μ tel que $q_\mu - 1$ est divisible par un nombre premier de la classe $\mu-1$ et ne contient d'autres diviseurs premiers que des nombres premiers des $\mu-1$ premières classes.

Le nombre de classes de M. Lipschitz peut surpasser tout nombre entier donné d'avance, car si q_μ est un nombre premier de la classe μ, tout nombre premier de la forme $hq_\mu+1$, où h est un nombre entier positif, appartiendra évidemment à une classe $\mu+k$ où $k>0$.

29.

Soient s un nombre premier quelconque et μ un nombre entier positif supérieur à un, on aura

$$\frac{x^{s^\mu}-1}{x^{s^{\mu-1}}-1} = x^{(s-1)s^{\mu-1}} + x^{(s-2)s^{\mu-1}} + \ldots\ldots + x^{s^{\mu-1}} + 1 \equiv s \pmod{x^{s^{\mu-1}}-1}$$

* J'ignore l'auteur de cette démonstration.

† Tout ce qui suit jusqu'à la fin de ce paragraphe a été ajouté après l'apparition du beau travail de M. Lipschitz (Journal für Mathematik, Bd. CV, p. 127-156).

quelle que soit la valeur entière de x. Il s'ensuit que tout diviseur premier q de $\frac{x^{s^\mu}-1}{x^{s^{\mu-1}}-1}$ autre que s est premier à $x^{s^{\mu-1}}-1$. On aura donc

$$x^{s^\mu} \equiv 1 \quad (\text{mod. } q),$$

mais non

$$x^{s^{\mu-1}} \equiv 1 \quad (\text{mod. } q)$$

ce qui fait voir que x appartient à l'exposant s^μ (mod. q). Le nombre $\phi(q) = q-1$ doit donc être divisible par s^μ et q est par conséquent de la forme $hs^\mu + 1$ où h est un nombre entier positif. Le nombre $\frac{s^{s^\mu}-1}{s^{s^{\mu-1}}-1}$ étant supérieur à l'unité admettra nécessairement un diviseur premier q_1 qui sera différent de s et par suite de la forme $hs^\mu + 1$. Le nombre $\frac{(q_1 s)^{s^\mu}-1}{(q_1 s)^{s^{\mu-1}}-1}$ admettra un diviseur premier q_2 différent de s et q_1 et par suite de la forme $hs^\mu + 1$. De même le nombre $\frac{(q_1 q_2 s)^{s^\mu}-1}{(q_1 q_2 s)^{s^{\mu-1}}-1}$ admettra un diviseur premier q_3 différent de s, q_1 et q_2 et par suite de la forme $hs^\mu + 1$. En continuant de la même manière on peut former autant de nombres premiers de la forme $hs^\mu + 1$ que l'on veut.

Soit maintenant q_1 un nombre premier de la forme $hs^\theta + 1$ obtenu par le procédé qu'on vient d'exposer, désignons par $A_1^{s^\theta}$ le groupe formé par l'ensemble des solutions de la congruence

$$x^{s^\theta} \equiv 1 \quad (\text{mod. } q_1).$$

Nous avons vu dans le §26 que le nombre des solutions de la congruence

$$x^s \equiv 1 \quad (\text{mod. } q_1)$$

est égal à s, il s'ensuit que le groupe $A_1^{s^\theta}$ est monobase et comme il résulte de la formation même du nombre premier q_1 que le groupe $A_1^{s^\theta}$ renferme au moins un élément appartenant à l'exposant s^θ, il est clair que l'ordre du groupe $A_1^{s^\theta}$ sera égal à s^θ. Formons d'une manière analogue un autre groupe monobase A_2^{s} d'ordre s^θ à l'aide d'un nombre premier q_2 différent de q_1 et ainsi de suite jusqu'à un certain groupe monobase $A_{m_\theta}^{s^\theta}$ d'ordre s^θ qu'on formera à l'aide d'un nombre premier q_{m_θ} différent des nombres premiers

$$q_1, q_2, q_3, \ldots q_{m_\theta - 1}.$$

Formons d'une manière analogue un certain nombre $m_{\theta-1} - m_\theta$ de groupes monobases

$$A_{m_\theta+1}^{s^{\theta-1}}, A_{m_\theta+2}^{s^{\theta-1}}, \ldots A_{m_\theta-1}^{s^{\theta-1}}$$

d'ordre $s^{\theta-1}$ à l'aide de nombres

$$q_{m_\theta+1},\ q_{m_\theta+2},\ \ldots\ q_{m_{\theta-1}}$$

différents tant entre eux que des nombres premiers

$$q_1,\ q_2,\ \ldots\ q_{m_\theta}.$$

On continuera la même opération jusqu'à ce qu'on aura formé $m_1 - m_2$ groupes monobases d'ordre s

$$A^s_{m_2+1},\ A^s_{m_2+2},\ \ldots\ A_{m_1}$$

à l'aide de nombres premiers

$$q_{m_2+1},\ q_{m_2+2},\ \ldots\ q_{m_1}$$

différents tant entre eux que des nombres premiers

$$q_1,\ q_2,\ \ldots\ q_{m_2}.$$

Je pose maintenant $$M = q_1 q_2 \ldots q_{m_1}$$

et je remplace tout élément a du groupe $A_1^{s^\theta}$ par un élément a' satisfaisant aux conditions

$$\begin{aligned} a' &\equiv a \quad (\mathrm{mod.}\ q_1), \\ a' &\equiv 1 \quad (\mathrm{mod.}\ q_2), \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \\ a' &\equiv 1 \quad (\mathrm{mod.}\ q_{m_1}), \\ a' &< M \end{aligned}$$

et de même pour les autres groupes. Cela ne change en rien la nature des groupes A et maintenant tous leurs éléments peuvent être considérés aussi comme faisant partie du groupe $\mathfrak{M}$ formé par l'ensemble de tous les nombres inférieurs à M et premiers à M. Encore plus, si l'on prend deux éléments a et b du groupe $A_1^{s^\theta}$, l'élément c qu'on obtient par la composition des éléments a et b correspondera à l'élément c' du groupe $\mathfrak{M}$, formé par la composition des éléments a' et b' de ce groupe qui correspondent respectivement aux éléments a et b du groupe $A_1^{s^\theta}$. Et de même pour les autres groupes. Cela étant ainsi, je désigne toujours par le signe $A_1^{s^\theta}$ le groupe $A_1^{s^\theta}$ ainsi transformé, et je dis que les groupes

$$A_1^{s^\theta},\ A_2^{s^\theta},\ \ldots\ A^s_{m_1}$$

sont susceptibles d'être combinés en un groupe Ξ_θ. En effet, si l'on désigne par

$$a_1,\ a_2,\ \ldots\ a_{m_1}$$

des éléments appartenant respectivement aux groupes

$$A_1^{s^\theta},\ A_2^{s^\theta},\ \ldots\ A_{m_1}^{s}$$

une congruence telle que

$$a_1 \equiv a_2 a_3 \ldots a_{m_1} \quad (\text{mod. } M)$$

ne peut avoir lieu à moins qu'on ait

$$a_1 \equiv 1 \quad (\text{mod. } M)$$

car on a

$$a_2 \equiv a_3 \equiv \ldots \equiv a_{m_1} \quad (\text{mod. } q_1).$$

En posant

$$\Xi_\theta = A_1^{s^\theta} A_2^{s^\theta} \ldots A_{m_1}^{s}$$

on aura un groupe tout à fait analogue à celui que nous avons étudié sous ce nom dans les paragraphes 7–10.

Soit maintenant E un groupe eulérien quelconque et que sa décomposition en groupes uniprimes d'ordres premiers entre eux, donne

$$E = \Xi_{\theta_1}\Xi_{\theta_2} \ldots \Xi_{\theta_n}$$

formons à l'aide de nombres

$$M_1,\ M_2,\ \ldots\ M_n$$

premiers deux à deux, des groupes

$$H_1,\ H_2,\ \ldots\ H_n$$

tout à fait analogues aux groupes

$$\Xi_{\theta_1},\ \Xi_{\theta_2},\ \ldots\ \Xi_{\theta_n}.$$

Supposons de plus qu'on ait choisi les éléments du groupe H_1 de manière qu'ils soient tous congrus à un suivant les modules M_2, M_3, ... M_n et de même pour les autres groupes, le groupe

$$F = H_1 H_2 \ldots H_n \quad (\text{mod. } M = M_1 M_2 \ldots M_n)$$

sera tout à fait analogue au groupe E.

La théorie des groupes eulériens trouve donc une application complète dans la théorie des restes suivant un module quelconque.

L'application du procédé du géomètre grec à d'autres cas de la proposition de Lejeune Dirichlet sur la progression arithmétique fera l'objet d'un prochain article.

GRA-THUMIAO, le 26 août 1889.

Ether Squirts.

Being an Attempt to specialize the form of Ether Motion which forms an Atom in a Theory propounded in former Papers.

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Résumé.

In three previous papers (Camb. Phil. Trans., Vol. XIV, page 71; London Math. Society's Proceedings, Vol. XX, p. 38 and p. 297) I have developed the results which flow from supposing the ultimate atom to be a sphere pulsating in a perfect fluid. I have shown that this hypothesis is not without suggestion for the phenomena of chemical affinity, cohesion, and spectrum analysis in the first paper; that it can be applied to explain dispersion and other optical phenomena, as well as certain electrical and magnetic phenomena in the second paper; while the fact that it leads to generalized elastic equations is developed in the third paper. In the present memoir I have endeavored to show that all these results still hold good if the pulsating sphere be replaced by an ether squirt which resists variations in its rate of flow. From whence the ether flows and why its flow resists variations are problems which, as they fall outside the range of physics, I leave to the metaphysicians to settle. The ether squirt as a model dynamic system for the atom seems at any rate to possess the property of simplicity. But the action of one group of ether squirts upon a second group leads to equations the complexity of which seems quite capable of paralleling any intricacy of actual Nature. The main portion of this paper is devoted to an investigation of inter-atomic and inter-molecular forces, and brings out the striking influence in producing cohesion of 'kin-atoms' in different molecules. Thus if A_2B and $A_2'B'$ represent two molecules compounded of two A atoms and one B atom, it is shown that the inter-atomic action of A on A in the first molecule is largely influenced by the action of A' in an adjacent molecule. In many cases the term which A''s action

produces in the atomic action of A on A is as important as that of A upon B. The effect of this action of kin-atoms in different molecules is shown to lead to the 'hypothesis of modified action' and to 'multi-constant' equations of elasticity. Its denial of the *literal* truth of the Second Law of Motion is also discussed, and its application to certain general problems of cohesion dealt with.

The law of gravitation and the theory of the potential are shown to be more intelligible on the ether squirt theory than on that of the pulsating sphere as developed in §§51–9 of my first paper.

On a Certain Atomic Hypothesis—Ether Squirts.

1). "It has become the fashion," said Professor Fitzgerald in his Bath address to the British Association, "to indulge in quaint cosmical theories and to dilate upon them before learned societies and in learned journals. I would suggest, as one who has been bogged in this quagmire, that a successor in this chair might well devote himself to a review of the cosmical theories propounded within the last few years. The opportunities for piquant criticism would be splendid."

The pleasure of "bogging oneself in this quagmire" is so great that even piquant criticism cannot restrain me from adding another quaint cosmical theory to the many that already exist. This hypothesis may be briefly summed up in the statement: *that an atom or the ultimate element of ponderable matter is an ether squirt.* (See Clifford's use of the term 'squirt,' *Elements of Dynamic*, I, p. 214.)

2). As it is well known, the vortex theory of matter reduces the ether to a perfect fluid, and endeavors to build up matter by some form of motion in this fluid. The infinite variety of motions which a perfect fluid is capable of, suggest all sorts of rotational or even irrotational forms which may account for matter. The great beauty of all such fluid motion solutions is their reduction of the physical universe to a single imponderable medium; they avoid dualistic explanations of natural phenomena. As, however, no sufficiently simple vortex origin of matter has yet been formulated, many scientists, notably Sir William Thomson, have, for the practical purpose of explaining optical phenomena, fallen back for the present on mechanical molecules. The mechanical molecule of Thomson, while of great assistance to the understanding of many facts of dispersion and absorption, undoubtedly possesses chemical disadvantages; it presents no obvious mode of disassociation. On the other hand, Lindemann has recently shown that it can

be used effectively to explain various phenomena of magnetism and electricity. He, however, assumes the ethereal medium to be of the nature of a *perfect fluid.* The Thomson-Lindemann atoms and molecules thus show us so far only complex mechanisms, and raise the not unnatural repugnance of the philosophical mind to a dualistic theory of the universe.

3). In a paper published in the Camb. Phil. Trans. (Vol. XIV, page 71), I have endeavored to explain certain optical and chemical phenomena by treating the ether as a perfect fluid and supposing the atom to be a differentiated part of the ether, which, as a first approximation, may be considered of spherical form and the surface of which is capable of pulsation. I did not venture upon any suggestion as to the nature of this differentiated portion of the ether. I merely suggested that it possibly might be explained as an ether vacuum with surface energy. All that I required was a surface in the ether approximately spherical and capable of pulsation. In a later paper I have shown that most of the results which flow from the Thomson-Lindemann atom, together with a ready explanation of disassociation, could be obtained from the pulsating atom. The molecule based upon it seemed to me to have distinct advantages over the complex mechanism of their spring-shell molecule. In the present paper I suggest a kinematic fluid origin for the pulsating spherical atom. The sort of fluid motion which I have chosen is an irrotational one, and therefore one capable of far more ready handling than a vortex-sponge system. The difficulty of course arises as to how rays of light are to be propagated by transverse vibration in such a fluid medium, but then that difficulty also occurs in the vortex theory when it treats the ether outside the vortex atom or matter as a perfect fluid. It may also be urged against Lindemann's arguments which largely depend on the perfect fluid motion of the medium outside the spring-shell atom. Perhaps the perfect fluid is turbulent.

At the same time it may be remarked that our theory does not absolutely require the ether to be a perfect fluid. It supposes only: (1) that its velocity at any point is the differential of a certain function ψ with regard to the direction of the velocity, or if u, v, w be the velocity components,

$$u = \frac{d\psi}{dx}, \quad v = \frac{d\psi}{dy}, \quad w = \frac{d\psi}{dz}, \tag{1}$$

(2) that the ether is incompressible and continuous except at points occupied by 'matter,' or

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \tag{2}$$

Lastly, it assumes: (3) that the kinetic energy of the *whole system* is given by the same expression as in the case of a perfect fluid; that is,

$$\text{Kinetic Energy} = -\tfrac{1}{2}\rho\int\psi\frac{d\psi}{dn}\,d\sigma, \tag{3}$$

where the integration is to be over the surface of all 'matter,' and n is the direction of the normal at the element $d\sigma$ of such surface, its positive sense being toward the fluid, ρ being the density of the medium. Whatever the nature of the ether, this is obviously an irrotational motion, but beyond the above assumptions it is not necessary to consider the ether as identical with a perfect fluid.

4). Now the results of the paper to which I have referred above were obtained by supposing the spherical atom to have certain surface pulsations; that is, the velocity of the fluid at the surface of the atom was supposed to be given and equal to the surface velocity of the particle of 'matter' there situated. Now we should have obtained precisely the same equations had we supposed a flow of the ether over the surface of the atom with the same velocity as that with which we endowed the atomic surface. Now this might arise from two causes, either (a) we may suppose a source or sink of ether inside the atom, the ether itself being incompressible; or (b) we may suppose the ether at the atom to be compressible, and the atom might be treated as an element of ether in the 'gaseous state,' a bubble of ether vapor in ether—just as we might have a bubble of the vapor of a substance in its liquid at certain temperatures and pressures. The difficulty of this latter explanation is really its dualistic character; it throws back the explanation of gas and liquid on something pre-ethereal and tends to suggest a second ether to explain the first.

On the other hand, our first hypothesis, especially when we simplify our atom by reducing it to a point, so that it really coincides with the source or sink, has a good deal to recommend it; it very much reduces the terms which appear in the expression for the kinetic energy of both atoms and ether; it retains, however, the terms that appear to be really necessary, and it renders these terms absolutely accurate instead of only approximations. It is, supposing it will suffice to explain physical phenomena, an extremely simple monistic hypothesis: the fluid medium in irrotational motion is the primary substance, the atom or element of matter is a squirt of the same substance. From whence the squirt comes into three-dimensioned space it is impossible to say; the theory limits our possibility of knowledge of the physical universe to the existence of the squirt. It

may be an argument for the existence of a space of higher dimensions than our own, but of that we can know nothing; and we are only concerned with the flow into our own medium, with the ether squirt which we propose to term 'matter.' Some idea of the theory may be obtained in the following manner: Suppose a perfectly smooth metal plate on which are placed any number of electrodes capable of free motion on the surface of the plate, then if these were sources and sinks for electricity passing in and out of the plate, they would move each other about, and in doing so they would move the really immaterial sinks and sources in the plate. To a being in the flatland provided by the electric medium of the plate, these really immaterial sinks and sources would appear to be centres which mutually accelerated each other, i. e. he would suppose them endowed with force, or, as seats of force, he would properly term them matter. The ether squirt in three-dimensioned space is endowed in the same way with the properties of 'matter.' It possesses, besides, all the singular merit of the Boscovichian atom, i. e. it is for theoretical purposes a mathematical point.

Of course we cannot understand what is the exact meaning of both the ether potential ψ and the ether velocity becoming infinite at the squirt or element of matter, but this difficulty would be a like puzzle to the dweller in the electric plate to which I have previously referred.

If the ether be incompressible, the total flow in and out at all squirts must be zero, or if this flow be proportional to the 'mass,' we find the total quantity of matter in the universe is constant, namely, *zero*. This would involve the existence of negative as well as positive matter in the universe, but the law of gravitation as deduced from our theory tells us that there must be a large preponderance of matter of like sign in that part of the universe with which we are acquainted. It is quite possible, however, that matter outside the solar system may be of a different sense, or even perhaps the ring of a planet be composed of matter of a different sense to the matter of the planet itself. The principle of the indestructibility of matter would reduce to the obvious theorem that we cannot stop the flow of any squirt, without a like flow coming into our space somewhere else—experience tells us, in the immediate neighborhood of the old squirt. We can vary the flow of any squirt and so obtain positive and negative electricity, but we cannot, so far as our present experience goes, create positive and negative matter—an ether source and a sink—at a point where there was previously no squirt at all.

5). In the following discussion the flow of any squirt will not be considered a constant, only its mean value over some long or short interval will be; thus the expression for the flow may contain periodic terms due either to variations characteristic of the nature of the particular squirt, or to variations forced upon it by the vibratory character of the flow of adjacent squirts. Thus if V_s be the velocity of flow of the ether just by the s^{th} squirt, we shall have

$$V_s = {}_0v_s + {}_1v_s \sin(n_1t + \alpha_1) + {}_2v_s \sin(n_2t + \alpha_2) + {}_3v_s \sin(n_3t + \alpha_3) + \text{etc.} \quad (4)$$

Here the constant term will be used to explain gravitation, the enforced periodic variations to throw light on chemical affinity and cohesion, while other vibrational terms will be suggestive for optical and electrical phenomena.

6). Let $r_1, r_2, r_3, \ldots . r_s, \ldots .$ be the distances of any point in the ether from the 1st, 2d, 3d, ... s^{th}, ... squirts, then a suitable solution of equations (1) and (2) is

$$\psi = \frac{A_1}{r_1} + \frac{A_2}{r_2} + \frac{A_3}{r_3} + \ldots . + \frac{A_s}{r_s} + \ldots . , \quad (5)$$

where $A_1, A_2, A_3, \ldots . A_s, \ldots .$ are any constants. To determine these, let $Q_1, Q_2, \ldots . Q_3, \ldots .$ be the instantaneous rates at which ether is flowing in at the 1st, 2d, ... s^{th}, ... squirts, then we have

$$Q_s = \int \frac{d\psi}{dr_s} r_s^2 d\omega = 4\pi r_s^2 V_s \rho,$$

where r_s is to be taken very small and the integration is to include all the solid angle round the s^{th} squirt. Since r_s^2 is to be taken very small, we find for the limit

$$Q_s = -4\pi A_s,$$

or,

$$\psi = -\frac{1}{4\pi\rho}\left\{\frac{Q_1}{r_1} + \frac{Q_2}{r_2} + \frac{Q_3}{r_3} + \ldots . + \frac{Q_s}{r_s} + \ldots .\right\}. \quad (6)$$

Now if Q_s were finite, V_s would become infinite when we make r_s vanishingly small. In order to avoid this, we shall assume V_s to be the radial velocity of the flow on the surface of a very small sphere of radius a_s surrounding the squirt, this sphere being taken so close that it only encloses its own squirt, and that the whole flow may be considered radial and uniform over its surface. We thus find

$$V_s = Q_s / 4\pi a_s^2 \rho. \quad (7$$

We proceed now to determine from (3) the entire kinetic energy of the system; we have, $d\omega$ being an element of solid angle,

$$K.E. = \frac{1}{8\pi}\Sigma\int\left(\frac{Q_1}{r_1}+\frac{Q_2}{r_2}+\ldots+\frac{Q_s}{r_s}+\ldots\right)$$
$$\times\frac{1}{4\pi\rho}\left(\frac{Q_1}{r_1^2}\frac{dr_1}{dr_s}+\frac{Q_2}{r_2^2}\frac{dr_2}{dr_s}+\ldots+\frac{Q_s}{r_s^2}+\ldots\right)r_s^2\,d\omega,$$

the summation extending over all values of s from 1 to ∞ and r_s being put in each case very small in the limit. Thus we have

$$\begin{aligned}K.E. &= \frac{1}{32\pi^2\rho}\Sigma Q_s\int\left(\frac{Q_1}{r_{1s}}+\frac{Q_2}{r_{2s}}+\ldots+\frac{Q_s}{r_s}+\ldots\right)d\omega\\ &= \frac{1}{8\pi\rho}\Sigma Q_s\left(\frac{Q_1}{r_{1s}}+\frac{Q_2}{r_{2s}}+\ldots+\frac{Q_{s-1}}{r_{s-1,s}}+\frac{Q_{s+1}}{r_{s+1,s}}+\ldots\right)+\frac{1}{32\pi^2}\Sigma\int\frac{Q_s^2}{r_s}\,d\omega\\ &= \frac{1}{4\pi\rho}\,\Sigma\,\frac{Q_sQ_{s'}}{\gamma_{ss'}}+\frac{1}{8\pi\rho}\Sigma\frac{Q_s^2}{a_s},\end{aligned}$$

where in the first summation s and s' are to take all possible different values and $\gamma_{ss'}$ is the distance between the s^{th} and s'^{th} squirts, and in the second summation we are to sum for all squirts. The terms of the last summation would become infinite if we did not exclude the volume of the fluid absolutely at the squirt. Substituting for Q_s in terms of V_s, we find

$$K.E. = \Sigma 2\pi\rho a_s^3 V_s^2 + \Sigma\,\frac{4\pi\rho a_s^2 a_{s'}^2}{\gamma_{ss'}}\,V_s V_{s'}. \tag{8}$$

This is the total energy of the ethereal medium excluding small spherical elements in the immediate neighborhood of the squirts. It agrees with the result obtained in Art. 14, p. 82, of the first paper on *pulsating* atoms (Camb. Phil. Trans., Vol. XIV) as a *first approximation* to the kinetic energy. It has therefore two advantages over the result given in that paper: it is no longer an approximation, but accurate so far as the hypothesis reaches; and farther, V_s need not be, like the $\dot{\phi}_0$ of the notation of that paper, merely periodic; it may contain a constant term, which would have involved rupture in the case of the pulsating spherical atom.

7). Let us deal first with the mean velocity of flow from the squirt across the surface of the small sphere of radius a_s. This we have represented by ${}_0v_s$. Since this does not vary with the time, it cannot arise in any equations—such as the Hamiltonian equations—deduced from the kinetic energy by varying the time. Hence it will not arise when we consider the forced variations in the flow

of the squirts. It contributes terms to the kinetic energy of the system depending upon the mutual distances of the squirts

$$= \Sigma \frac{4\pi\rho a_s^2 a_{s'}^2 {}_0v_s {}_0v_{s'}}{\gamma_{ss'}}$$

$$= \frac{1}{4\pi\rho} \Sigma \frac{q_s q_{s'}}{\gamma_{ss'}},$$

if q_s be the mean rate at which ether is poured in at the s^{th} squirt.

Now let us define 'mass' to be the "mean rate at which ether is poured in at any squirt." Then we have the following result:

There is an attractive force between any one element of matter and any other element of matter, varying directly as their masses and inversely as the square of their distance.

This is the ordinary law of gravitation. The constant $1/4\pi\rho$ may either be taken as a constant of attraction or in the form $\sqrt{1/4\pi\rho}$ as a factor of mass.

If we write $\sqrt{1/4\pi\rho}q_s = m_s$, we have, confining our attention to one element of matter m_s, the following terms in the kinetic energy:

$$m_s \Sigma \frac{m_{s'}}{\gamma_{ss'}}.$$

Now this element of the kinetic energy of the ether appears as potential energy when we leave the ether out of sight—thus, the potential energy of gravitating bodies is disguised kinetic energy of the ether. The potential of a number of elementary masses at any point not occupied by one of them is of the form

$$\frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_3}{r_3} + \ldots,$$

or remembering the values of m_1, m_2, etc., we see that this is equal to the *mean* value of $-\psi\sqrt{4\pi\rho}$ for the same set of elementary masses: see equation (6). Thus the gravitational potential differs only by a change in sign and a constant factor from the velocity potential of the ether at the same point. It thus obtains a definite physical meaning with regard to the ether. Further, we note that if an element of matter (or a squirt) be placed at any point of a field in which there are other gravitating elements, their effect on the new element is to cause it to move along the line of flow of the ether at that point due to the gravitating system; that it moves with an acceleration proportional to the speed of the ether flow but in an *opposite* sense. Most of the properties of the potential are thus capable

of being stated in language involving either the kinetic energy or local motion of the ether.

8). Leaving the gravitational terms, let us endeavor to form equations for the forced vibrations in a set of ether squirts. Here a difficulty arises at the very outset, owing to our ignorance of why and how the ether is injected into the ethereal medium. Let us suppose p_s to represent the amount of ether injected from a given epoch up to the instant in question through the s^{th} squirt, then obviously the time variation of p_s, or $\dot{p}_s$, is what we have written Q_s. Thus the kinetic energy may be written in the form

$$K.E. = \frac{1}{8\pi\rho}\Sigma\frac{\dot{p}_s^2}{a_s} + \frac{1}{4\pi\rho}\Sigma\frac{\dot{p}_s\dot{p}_{s'}}{\gamma_{ss'}}. \tag{9}$$

Now the equations for the forced vibrations in the flow of the squirt will be meaningless unless the squirt tends to *resist* variations in its rate of flow. But we do not know why it flows, much less the reason why only a limited variation is permissible in its rate of flow. That depends on the state of affairs outside the space which is alone sensible to us and with which we can deal. We can, as it were, only guess at the 'potential energy' of our atom, which lies outside our space. On the ether squirt hypothesis, the mechanism of the *Ding-an-sich* is beyond our control or inspection. One thing, however, is obvious, if the theory is to apply to facts as we understand them, only a slight change in the rate of flow is possible, and the phenomena of the spectrum, especially on the "single bright line" theory of an absolutely disassociated atom, lead us to believe that this variation is periodic and characteristic for each individual atom. This suggests and in part justifies our adding to the total energy of the system a potential energy of the form

$$\tfrac{1}{2}\Sigma\tau_s'(p_s - q_s t)^2.$$

For this gives a 'force' at the s^{th} squirt tending to *resist* variation of its rate of flow equal to

$$\tau_s'(p_s - q_s t),$$

or proportional to the amount which the forced vibration has drawn from the ether-store of the particular squirt over and above its ordinary gravitation allowance. We have termed this 'potential energy,' but just as we have banished potential energy by our hypothesis from the space under our control, so we might replace this by kinetic energy if we were able to include the unknown system outside our space.

Thus the difference L of the kinetic and potential energies of the system, supposing *there to be no motion of translation*, is given by

$$L = \frac{1}{8\pi\rho}\Sigma\frac{\dot{p}_s^2}{a_s} + \frac{1}{4\pi\rho}\Sigma\frac{\dot{p}_s\dot{p}_{s'}}{\gamma_{ss'}} - \tfrac{1}{2}\Sigma\tau_s'(p_s - q_s t)^2.$$

Hence, applying Lagrange's equations, we obtain for the typical vibrational equation

$$\frac{1}{4\pi\rho a_s}\ddot{p}_s + \frac{1}{4\pi\rho}\Sigma\frac{\ddot{p}_{s'}}{\gamma_{ss'}} + \tau_s'(p_s - q_s t) = 0.$$

Let $\left.\begin{array}{l}\phi_s = p - q_s t\\ \tau_s = 4\pi\rho\tau_s'\end{array}\right\}$, then we have

$$\ddot{\phi}_s/a_s + \Sigma\ddot{\phi}_{s'}/\gamma_{ss'} + \tau_s\phi_s = 0. \tag{10}$$

This is practically identical with the typical equation by means of which I have endeavored, in Arts. 14–25 (pp. 82–92) of my first paper, to represent the main facts of spectrum analysis. It is also the equation which leads on a fairly plausible hypothesis to a formula for the dispersion of light passing through a medium composed of such ether squirts exactly similar to that found by Thomson and Lindemann in the case of a spring-shell molecule (see my second paper, London Math. Soc. Proc., Vol. XX, p. 41). So far, then, the ether squirt is equally efficient with the pulsating spherical atom (or, for the matter of that, with the spring-shell molecule) in explaining optical phenomena. It is obviously much more suggestive for the problem of gravitation. We must turn now to its bearing on atomic and cohesive forces.

9). *To find the atomic force between the s^{th} and s'^{th} atoms of a group of k atoms forming a molecule.*

Looking at equation (10), we observe that if two atoms were at a distance which was great compared with the radii of the small spheres by which we have enclosed each squirt (we may take these radii equal if we please; they are not 'atomic radii,' for our atom is really a Boscovichian point), then the summation term would vanish as compared with the other, and the flow of the squirt vary with its own characteristic period $2\pi/\sqrt{\tau_s a_s}$, or $2\pi/\nu_s$ say. Hence in this case

$$p_s = q_s t + C_s \cos(\nu_s t + \alpha_s).$$

We have now to solve k equations of the type (10) which may be written

$$\ddot{\phi}_s + \Sigma\ddot{\phi}_{s'}\frac{a_s}{\gamma_{ss'}} + \nu_s^2\phi_s = 0. \tag{11}$$

Assume $\phi_s = B_s \cos(nt + \beta)$, then

$$B_s(\nu_s^2 - n^2) - \Sigma B_{s'} n^2 \frac{a_s}{\gamma_{ss'}} = 0,$$

or

$$B_s\left(\frac{1}{\nu_s^2} - \frac{1}{n^2}\right) + \frac{1}{\nu_s^2} \Sigma B_{s'} \frac{a_s}{\gamma_{ss'}} = 0. \tag{12}$$

This gives to determine $\frac{1}{n^2}$ the determinantal equation

$$\begin{vmatrix} 1 - \frac{\nu_1^2}{n^2}, & \frac{a_2}{\gamma_{12}}, & \frac{a_3}{\gamma_{13}}, \ldots & \frac{a_k}{\gamma_{1k}} \\ \frac{a_1}{\gamma_{12}}, & 1 - \frac{\nu_2^2}{n^2}, & \frac{a_3}{\gamma_{23}}, \ldots & \frac{a_k}{\gamma_{2k}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{a_1}{\gamma_{1k}}, & \frac{a_2}{\gamma_{2k}}, & \ldots\ldots & 1 - \frac{\nu_k^2}{n^2} \end{vmatrix} = 0. \tag{13}$$

The roots of n obtained from this equation substituted in (12) give the constants B_s. In order to obtain some idea of the nature of the periods and the constants we may find approximations to their value in series ascending by powers of a/γ. We easily find as a first approximation to the s^{th} root of the determinant

$$\frac{1}{n_s^2} = \frac{1}{\nu_s^2} - \Sigma \frac{a_s a_{s'}}{\gamma^2_{ss'}} \frac{1}{\nu_s^2 - \nu_{s'}^2}, \tag{14}$$

the summation extending to all values of s' other than s. Further, if B_s represent the amplitude of the s^{th} periodic vibration in the s^{th} atom, we find

$$B_{s'} = -\frac{B_s \nu_s^2}{\nu_s^2 - \nu_{s'}^2} \frac{a_{s'}}{\gamma_{ss'}}. \tag{15}$$

Thus we have

$$\phi_{s'} = B_{s'} \cos(n_{s'} t + \beta_{s'}) - \Sigma \frac{B_s \nu_s^2}{\nu_s^2 - \nu_{s'}^2} \frac{a_{s'}}{\gamma_{ss'}} \cos(n_s t + \beta_s), \tag{16}$$

the summation extending to all values of s other than s'. Thus finally we have

$$\dot{p}_{s'} = q_{s'} - n_{s'} B_{s'} \sin(n_{s'} t + \beta_{s'}) + \overset{(s \text{ all values but } s')}{\Sigma} \frac{n_s B_s \nu_s^2}{\nu_s^2 - \nu_{s'}^2} \frac{a_{s'}}{\gamma_{ss'}} \sin(n_s t + \beta_s),$$

$$\dot{p}_s = q - n_s B_s \sin(n_s t + \beta_s) + \overset{(s' \text{ all values but } s)}{\Sigma} \frac{n_{s'} B_{s'} \nu_{s'}^2}{\nu_{s'}^2 - \nu_s^2} \frac{a_s}{\gamma_{ss'}} \sin(n_{s'} t + \beta_{s'}).$$

We must now return to equation (9) and substitute these values of $\dot{p}_s$ and $\dot{p}_{s'}$ in the term which involves $1/\gamma_{ss'}$. This will give us the *apparent* force function U between the atoms. In order to ascertain its measurable value we must take its mean with regard to an interval of time embracing many vibrations, we easily find

$$\text{mean of } \dot{p}_{s'}\dot{p}_s = q_{s'}q_s - \tfrac{1}{2}\frac{n_s^2\nu_s^2B_s^2}{\nu_s^2-\nu_{s'}^2}\frac{a_{s'}}{\gamma_{ss'}} - \tfrac{1}{2}\frac{n_{s'}^2\nu_{s'}^2B_{s'}^2}{\nu_{s'}^2-\nu_s^2}\frac{a_s}{\gamma_{ss'}} + \tfrac{1}{2}\underset{(r \text{ all values but } s \text{ or } s')}{\Sigma}\frac{n_r^2B_r^2\nu_r^4}{(\nu_r^2-\nu_{s'}^2)(\nu_r^2-\nu_s^2)}\frac{a_sa_{s'}}{\gamma_{rs'}\gamma_{rs}}.$$

The first term here, $q_{s'}q_s$, we may omit; it gives rise only to the gravitation potential with which we have already dealt. Retaining terms only as far as the cube of a/γ, we have

$$\begin{aligned}\text{mean of } \dot{p}_s\dot{p}_{s'} = {} & \tfrac{1}{2}\frac{a_{s'}\nu_s^4B_s^2 - a_{s'}\nu_{s'}^4B_{s'}^2}{\nu_{s'}^2-\nu_s^2}\frac{1}{\gamma_{ss'}} \\ & + \tfrac{1}{2}\frac{a_{s'}\nu_s^6B_s^2 + a_s\nu_{s'}^6B_{s'}^2}{(\nu_s^2-\nu_{s'}^2)^2}\frac{a_sa_{s'}}{\gamma_{ss'}^3} \\ & + \tfrac{1}{2}\underset{(r \text{ all values but } s \text{ or } s')}{\Sigma}\frac{\nu_r^6B_r^2}{(\nu_r^2-\nu_{s'}^2)(\nu_r^2-\nu_s^2)}\frac{a_sa_{s'}}{\gamma_{rs'}\gamma_{rs}}.\end{aligned}$$

Thus for the *attractive* atomic force between the s^{th} and s'^{th} atom we have, from $F_{ss'} = -dU/d\gamma_{ss'}$,

$$\left.\begin{aligned}F_{ss'} = {} & \frac{a_{s'}\nu_s^4B_s^2 - a_s\nu_{s'}^4B_{s'}^2}{4\pi\rho(\nu_{s'}^2-\nu_s^2)}\frac{1}{\gamma_{ss'}^3} \\ & + \frac{a_{s'}\nu_s^6B_s^2 + a_s\nu_{s'}^6B_{s'}^2}{2\pi\rho(\nu_s^2-\nu_{s'}^2)^2}\frac{a_sa_{s'}}{\gamma_{ss'}^5} \\ & + \frac{1}{8\pi\rho}\Sigma\frac{\nu_r^6B_r^2}{(\nu_r^2-\nu_{s'}^2)(\nu_r^2-\nu_s^2)}\frac{a_sa_{s'}}{\gamma_{ss'}^3\gamma_{rs}\gamma_{rs'}}\end{aligned}\right\}. \tag{17}$$

This is the value of the inter-atomic force so far as the fifth power of the inverse of the atomic distance. Thus we find—

(i). Atomic force varies partially as the inverse cube, partially as the inverse fifth power of the distance.

(ii). The modifying action of other atoms of the same molecule (or of atoms within the range of chemical action) varies as the inverse *fourth* power of atomic distance, but only as the inverse *square* of the particular atomic distance of the two atoms between which we are measuring the force.

These results are in complete agreement with those obtained for *Atomic Forces* in Arts. 31–41 (pp. 96–104) of my first paper. The 'chemical intensity' and the 'chemical affinity' may be defined in precisely the same manner as in Art. 38 and the same explanations given of the laws of chemical combination and disassociation. Squirts act for atomic forces exactly as the pulsating spheres dealt with in that paper.

10). Some general remarks may, however, be made on the formula (17) for inter-atomic force. I have in this paper proceeded to a higher approximation than in Art. 37 of the first paper, and we see it introduces a term into the force between the s^{th} and s'^{th} atom which is next in importance to the cubic term and depends upon the distances of those atoms from the r^{th} atom. Thus what Jellett has termed the "hypothesis of modified action" holds for a system of ether squirt atoms. Now this warns us of the care which must be used in applying the literal interpretations of the definition of Force and of the Second Law of Motion to atoms. Let us examine this somewhat more closely.

Force is any cause which tends to alter a body's natural state of rest, or of uniform motion in a straight line.

Now the s'^{th} atom is a cause of such change of motion in the s^{th} atom. It exerts a 'force' upon it depending upon their relative positions.

Now the *Second Law* of motion is often stated in the following manner:

Whenever any forces whatever act on a body, then, whether the body be originally at rest or moving with any velocity and in any direction, each force produces in the body the exact change of motion which it would have produced if it had acted singly on the body originally at rest.

It is the last phrase of this law which gives rise to doubt. Force being defined as the cause of motion, it might be supposed from the law that we could superpose causes of motion without altering the effects they produce on the motion of the body when they act alone. But this is not true when we deal with atoms within the range of chemical action. The introduction of a third atom into a field containing two others not only introduces new forces between those two atoms, but profoundly modifies the force existing between the two first atoms. This was an impossibility so long as force was considered as something *inherent in matter* and the Second Law of Motion was intelligible in its literal sense, but the moment we throw back force on the kinetic energy of the ethereal medium, then the force between any two elements is dependent not

solely on the atoms but on the whole ethereal field. The true way out of the difficulty is to disregard the second law and treat the instantaneous acceleration of any element as a function of its position with regard to the whole of the surrounding field. Nor is this alone sufficient; the force cannot in another sense be identified with the cause of change in motion. The cause of acceleration of a particular atom in the ether depends not only on its position in the field, but also *on its own motion*, on its own periodic translatory and internal vibratory motions as well as on the similar motions of other atoms. Hence, if force be identified with the 'cause of change in motion,' it does not produce the same change when we place the atom in a state of complete rest. In fact, if we once got an atom into a state of complete rest, it would denote complete rest of the ether in its immediate neighborhood, and thus no force could or would be 'acting upon it.' In fact there is little doubt that the atom would have ceased to be when thus brought to rest. There seems to me therefore considerable danger in the literal application of Newton's definition of force and his laws of motion to the mutual acceleration of atoms. They seem to exclude the 'hypothesis of modified action' as well as the existence of a generalized strain-energy, both of which I hope to show in a future paper arise from inter-molecular force involving the speed of the molecules; such terms in inter-molecular force are really due to the kinetic energy of the ether. (See my third paper, London Math. Soc. Proceedings, Vol. XX, p. 297.)

11). Returning to formula (17), I need not rediscuss the possibilities it offers for explaining chemical actions, but I will draw attention to a new range of phenomena which possibly the introduction of the modifying term will throw some light upon. We will not consider the action of a third or fourth atom brought close to a molecule in breaking it up as in the first paper, but we will start with a molecule of k atoms and ask what physical changes can disassociate it. The force as the inverse fifth of the inter-atomic distance is always attractive, but it may be considered small as compared with the inverse cube term. On the other hand, a single term of the modifying force, while it varies as the inverse square of the distance, is yet of the inverse fourth order in mean inter-atomic distance. There are, however, any number of such terms, so that it may rise to equal importance with the cubic term if we take a sufficient number of atoms. Further, it has for its numerator a simple square of an amplitude, while the cubic term has the *difference* of two quantities involving the amplitudes squared. On both of these counts, then, they cannot in many cases be so widely different in

magnitude, and slight variations in their relative magnitude may assist us in explaining various phenomena. I take two theoretical examples.

(i). A and B are two atoms whose 'chemical affinity' (see Art. 38 of my first paper) is negative. They cannot enter into a chemical combination. This means that the coefficient of the cubic term is negative. If their 'antipathy' for each other be not very great, a third atom C (the 'r^{th} atom') may bring them into chemical union. For this purpose it is necessary that the modifying term should be positive and greater than the cubic term. In order that it should be positive, however, we must have v_r either greater or less than *both* v_s and $v_{s'}$, or the single bright line of the atom which would link A and B together must not lie between the characteristic bright lines of A and B. On the other hand, if the 'antipathy' of A to B is somewhat greater we may require two, three or more atoms to hold them together by their modifying action on A and B's inter-atomic force. Thus ABC might be an impossible chemical union, but ABC_2 or ABC_3 possible. Further, since the amplitudes B_s, $B_{s'}$, B_r, etc., are probably functions of the surrounding physical conditions, as pressure, temperature, etc., it follows that under certain physical conditions ABC_3 might be possible and ABC_2 not, etc., etc.

The disassociating effect of a third atom C on A and B when the chemical affinity of the latter is positive, may be discussed in like manner. In this case C's bright line must lie between those of A and B. Of course A and B, although they may now repel each other, may be still held together by the affinity of both for C, but this chemical combination will have very different physical properties from one in which A and B have themselves affinity.

(ii). Suppose the atoms A, B, C in chemical union, then if the modifying term be negative and the cubic term (between A and B) be positive, it is conceivable that a change in the physical condition—especially a change in the temperature—without the introduction of any further atoms, will invert the order of magnitude or produce an antipathy between A and B. We may thus state the principle: If three atoms, A, B, C, are in chemical union, and A have a real chemical affinity for B, this affinity will be strengthened by the presence of the atom C if its bright line does not lie between the characteristic bright lines of A and B, but the affinity on the other hand will be weakened if the line does lie between those of A and B. Anything that tends to excite C or to cause its amplitude to vary more rapidly than those of A and

B, e. g. a wave of light or heat nearly of the period of its characteristic bright line—will thus in the corresponding cases largely strengthen or weaken the affinity of A for B. It may produce a more stable compound or it may lead to disassociation. Thus the effect of heat on a chemical compound receives considerable light from the existence of the modified action term.

12). We now pass to molecular force, or to the force between groups of atoms constituting two separate molecules. The peculiarity of such groups is that the flow variation of an atom of one group is practically only affected by the atoms of its own group. If the atoms of one group are essentially affected by the atoms of another, then the groups really enter into a sort of partial chemical combination with each other. This, which is hardly a possibility in the case of a rare gas, might, on the other hand, be to some extent true for a solid or liquid body. The cohesive forces might be to some extent chemical forces. Possibly in certain loose compound substances molecules may be chemically compounded, but the drift of experiment seems to be against this sort of union as a rule; inter-molecular distance seems to be considerable as compared with inter-atomic distance, and the force between molecules to be thus of a different nature to that between atoms.* At the same time the phenomena of fluted and continuous spectra warn us that it is quite possible that for a full explanation of even inter-molecular action we may be thrown back on something of the nature of a chemical action between molecules.

Let us suppose we have three molecules of the same substance each containing k atoms, and let r, r' and r'' denote any triplet of corresponding atoms of the three. We have first to determine how far the vibrations of any one atom are influenced by the existence of the other two groups of atoms. To do this we have to find the $3k$ periods given by the characteristic determinant like (13), but in this special case every three roots of the determinant corresponding to the characteristic periods of the three like atoms from each molecule will be nearly equal. Let $\nu_1^2/a_1 = \tau_1$, $\nu_2^2/a_2 = \tau_2$, etc. Then we have to solve the determinantal equation

$$0 = \begin{vmatrix} \tau_1\left(\dfrac{1}{\nu_1^2} - \dfrac{1}{n^2}\right), & \dfrac{1}{\gamma_{1,1'}}, & \cdots\cdots & \dfrac{1}{\gamma_{1,3k}} \\ \dfrac{1}{\gamma_{1,1'}}, & \tau_2\left(\dfrac{1}{\nu_{1'}^2} - \dfrac{1}{n^2}\right), & \cdots\cdots & \dfrac{1}{\gamma_{2,3k}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{1}{\gamma_{1,3k}}, & \dfrac{1}{\gamma_{2,3k}}, & & \tau_{3k}\left(\dfrac{1}{\nu_{3k}^2} - \dfrac{1}{n^2}\right) \end{vmatrix}.$$

* See on this point W. Ramsay, Phil. Mag., 1887, p. 129, and his reference to van der Waal's views on the subject.

Here, if ν_1, ν_1', ν_1'' are the three equal constants for three like atoms in the three separate molecules, we require to find the values of n which differ only by a small quantity from ν_1. Let $1/\nu_1^2 - 1/n^2 = \mu$. Then, neglecting μ^4, we have to find μ from the equation

$$\begin{vmatrix} \tau_1\mu, & 1/\gamma_{1,1'}, & & \dots\dots 1/\gamma_{1,3k} \\ 1/\gamma_{1,1'}, & \tau_{1'}\mu, & & \dots\dots 1/\gamma_{2,3k} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 1/\gamma_{1,3k-1}, & 1/\gamma_{2,3k-1}, & \dots\dots \tau_{3k-1}\left(\frac{1}{\nu_{3k-1}^2} - \frac{1}{\nu_1^2}\right), & 1/\gamma_{3k-1,3k} \\ 1/\gamma_{1,3k}, & 1/\gamma_{2,3k}, & \dots\dots 1/\gamma_{3k-1,3k}, & \tau_{3k}\left(\frac{1}{\nu_{3k}^2} - \frac{1}{\nu_1^2}\right) \end{vmatrix} = 0. \tag{18}$$

The problem thus reduces to the following one in pure mathematics: To expand the following *symmetrical* determinant in powers of small quantities, where each constituent $a_{r,s}$ is small if r and s are different, but only $a_{n,n}, a_{n-1,n-1}, a_{n-2,n-2}$ of the diagonal constituents are also small,

$$\Delta = \begin{vmatrix} a_{n,n}, & a_{n,n-1}, & \dots\dots a_{n,1} \\ a_{n-1,n}, & a_{n-1,n-1}, & \dots\dots a_{n-1,1} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ a_{1,n}, & a_{1,n-1}, & a_{1,1} \end{vmatrix} \tag{19}$$

Now

$$\Delta = D_1 a_{n,n} - \sum_r^{1 \text{ to } n-1} a_{n,r}^2 \beta_{r,r} - 2\sum_{rs}^{1 \text{ to } n-1} a_{n,r} a_{n,s} \beta_{r,s}, \tag{20}$$

where D_1 is the determinant obtained by suppressing the first column and row and $\beta_{r,s}$ is any minor of D_1; r being different from s and all values from 1 to $n-1$ being given to them.

Similarly if D_2 be obtained from D_1 as D_1 from Δ and so on, we have

$$D_1 = D_2 a_{n-1,n-1} - \sum_r^{1 \text{ to } n-2} a_{n-1,r}^2 \beta_{r,r}' - 2\sum_{rs}^{1 \text{ to } n-2} a_{n-1,r} a_{n-1,s} \beta_{r,s}',$$

$$D_2 = D_3 a_{n-2,n-2} - \sum_r^{1 \text{ to } n-3} a_{n-2,r}^2 \beta_{r,r}'' - 2\sum_{rs}^{1 \text{ to } n-3} a_{n-2,r} a_{n-2,s} \beta_{r,s}''.$$

Now D_3 will occur in Δ multiplied by $a_{n,n} a_{n-1,n-1} a_{n-2,n-2}$ or $\tau_1\tau_{1'}\tau_{1''}\mu^3$; that is to say, if we keep only terms of the order μ^3 we may put for D_3 its diagonal $a_{n-3,n-3} \dots a_{1,1}$, which is the one term in it containing no further small quantities; we thus find to our order of approximation,

$$D_2 = a_{n-2,n-2}a_{n-3,n-3}\ldots\ldots a_{1,1} - \sum_r^{1 \text{ to } n-3} a^2_{n-2,r}\beta''_{r,r} - 2\sum_{rs}^{1 \text{ to } n-3} a_{n-2,r}a_{n-2,s}\beta''_{r,s}.$$

For brevity we shall write $a_{r,r}\ldots\ldots a_{1,1} = P(r,r)$. Now it is easy to see that the most important term in $\beta''_{r,s} = (-1)^{r+s}\dfrac{P(n-3,\,n-3)}{a_{r,r}a_{s,s}} \times a_{r,s}$ and thus the last summation appears in D_2 with terms of the cubic order of small quantities or in Δ with those of the fifth order, hence we neglect it.

Further, $\beta''_{r,r} = \dfrac{P(n-3,\,n-3)}{a_{r,r}}$ to our degree of approximation, thus finally,

$$D_2 = P(n-2,\,n-2) - P(n-3,\,n-3)\sum_r^{1 \text{ to } n-3}\frac{a^2_{n-2,r}}{a_{r,r}}.$$

We now turn to D_1 whose value must be taken to one degree higher approximation than D_2, we find by similar reasoning

$$\begin{aligned}
D_1 = D_2 \times a_{n-1,n-1} &- \sum_r^{1 \text{ to } n-3} a^2_{n-1,r}\frac{P(n-3,\,n-3)}{a_{r,r}} \times a_{n-2,n-2} \\
&- a^2_{n-1,n-2}P(n-3,\,n-3) - 2(-1)^{n-3-s}\sum_s^{1 \text{ to } n-3} a_{n-1,n-2}a_{n-1,s} \\
&\qquad \times a_{n-2,s}\frac{P(n-3,\,n-3)}{a_{s,s}} \\
= P(n-3,\,n-3)\Bigg\{ & a_{n-1,n-1}a_{n-2,n-2} - a_{n-1,n-1}\sum_r^{1 \text{ to } n-3}\frac{a^2_{n-2,r}}{a_{r,r}} - a^2_{n-1,n-2} \\
&- a_{n-2,n-2}\sum_r^{1 \text{ to } n-3}\frac{a^2_{n-1,r}}{a_{r,r}} - 2(-1)^{n-3-s}a_{n-1,n-2}\sum_s^{1 \text{ to } n-3}\frac{a_{n-1,s}a_{n-2,s}}{a_{s,s}}\Bigg\}.
\end{aligned}$$

Finally we obtain in like manner

$$\begin{aligned}
D = D_1 \times a_{n,n} &- \sum_r^{1 \text{ to } n-3}\frac{a^2_{n,r}}{a_{r,r}}P(n-3,\,n-3)\,a_{n-1,n-1}a_{n-2,n-2} \\
&- a^2_{n,n-1}a_{n-2,n-2}P(n-3,\,n-3) \\
&- a^2_{n,n-2}a_{n-1,n-1}P(n-3,\,n-3) \\
&+ 2a_{n,n-1}a_{n,n-2}a_{n-1,n-2}P(n-3,\,n-3) \\
&- 2(-1)^{n-3-s}\sum_s^{1 \text{ to } n-3}\frac{a_{n,n-1}a_{n,s}a_{n-1,s}}{a_{s,s}}P(n-3,\,n-3)\,a_{n-2,n-2} \\
&- 2(-1)^{n-3-s}\sum_s^{1 \text{ to } n-3}\frac{a_{n,n-2}a_{n,s}a_{n-2,s}}{a_{s,s}}P(n-3,\,n-3)\,a_{n-1,n-1}.
\end{aligned}$$

Thus we obtain for Δ the value

$$\frac{\Delta}{P(n-3,\, n-3)} = a_{n,n}\, a_{n-2,n-2}\, a_{n-1,n-1} - a_{n,n}\, a^2_{n-1,n-2} - a_{n-1,n-1}\, a^2_{n,n-2}$$

$$- a_{n-2,n-2}\, a^2_{n,n-1} + 2a_{n,n-1}\, a_{n,n-2}\, a_{n-1,n-2}$$

$$- a_{n-1,n-1}\, a_{n-2,n-2} \sum_r^{1 \text{ to } n-3} \frac{a^2_{n,r}}{a_{r,r}}$$

$$- a_{n,n}\, a_{n-1,n-1} \sum_r^{1 \text{ to } n-3} \frac{a^2_{n-2,r}}{a_{r,r}}$$

$$- a_{n-2,n-2}\, a_{n,n} \sum_r^{1 \text{ to } n-3} \frac{a^2_{n-1,r}}{a_{r,r}}$$

$$- 2(-1)^{n-3-s} a_{n,n}\, a_{n-1,n-2} \sum_s^{1 \text{ to } n-3} \frac{a_{n-1,s}\, a_{n-2,s}}{a_{s,s}}$$

$$- 2(-1)^{n-3-s} a_{n-1,n-1}\, a_{n,n-2} \sum_s^{1 \text{ to } n-3} \frac{a_{n,s}\, a_{n-2,s}}{a_{s,s}}$$

$$- 2(-1)^{n-3-s} a_{n-2,n-2}\, a_{n,n-1} \sum_s^{1 \text{ to } n-3} \frac{a_{n,s}\, a_{n-1,s}}{a_{s,s}}.$$

13). Resuming the notation of our molecular problem, we have to find μ the cubic:

$$\tau_1 \tau_{1'} \tau_{1''} \mu^3 - \mu^2 \left\{ \tau_{1'}\tau_{1''} \Sigma \frac{\tau_r^{-1}}{\gamma^2_{1r}} \left(\frac{1}{\nu_r^2} - \frac{1}{\nu_1^2} \right)^{-1} + \tau_1 \tau_{1'} \Sigma \frac{\tau_r^{-1}}{\gamma^2_{1''r}} \left(\frac{1}{\nu_r^2} - \frac{1}{\nu_1^2} \right)^{-1} \right.$$

$$\left. + \tau_1 \tau_{1''} \Sigma \frac{\tau_r^{-1}}{\gamma^2_{1'r}} \left(\frac{1}{\nu_r^2} - \frac{1}{\nu_1^2} \right)^{-1} \right\}$$

$$- \mu \left\{ \tau_1 \frac{1}{\gamma^2_{1'1''}} + \tau_{1'} \frac{1}{\gamma^2_{11''}} + \tau_{1''} \frac{1}{\gamma^2_{11'}} \right.$$

$$+ 2(-1)^p \frac{\tau_1}{\gamma_{1'1''}} \Sigma \frac{\tau_r}{\gamma_{1'r}\gamma_{1''r}} \left(\frac{1}{\nu_r^2} - \frac{1}{\nu_1^2} \right)^{-1}$$

$$+ 2(-1)^p \frac{\tau_1}{\gamma_{11''}} \Sigma \frac{\tau_r}{\gamma_{1r}\gamma_{1''r}} \left(\frac{1}{\nu_r^2} - \frac{1}{\nu_1^2} \right)^{-1}$$

$$\left. + 2(-1)^p \frac{\tau_{1''}}{\gamma_{11'}} \Sigma \frac{\tau_r}{\gamma_{1'r}\gamma_{1r}} \left(\frac{1}{\nu_r^2} - \frac{1}{\nu_1^2} \right)^{-1} \right\}$$

$$+ 2 \frac{1}{\gamma_{11'}\gamma_{11''}\gamma_{1'1''}} = 0.$$

Here the summations are to extend over all the atoms of all three molecules except the three equal atoms, one in each molecule which have $2\pi/\nu_1$ for their characteristic free period, and p equals the number of places from the s^{th} atom to the 2^{d} in the series

$$1,\ 1',\ 1'',\ 2,\ 2',\ 2'',\ \ldots .\ s,\ s',\ s'',\ \ldots .\ k,\ k',\ k''.$$

Remembering the value of τ_r, we may rewrite this equation,

$$\left.\begin{aligned}
&\mu^3-\mu^2\Sigma\left\{\frac{1}{\nu_1^2-\nu_r^2}\left(\frac{a_1a_r}{\gamma_{1r}^2}+\frac{a_{1'}a_r}{\gamma_{1'r}^2}+\frac{a_{1''}a_r}{\gamma_{1''r}^2}\right)\right\}\\
&\quad-\mu\left\{\frac{1}{\nu_1^4}\left(\frac{a_{1'}a_{1''}}{\gamma_{1'1''}^2}+\frac{a_1a_{1''}}{\gamma_{11''}^2}+\frac{a_1a_{1'}}{\gamma_{11'}^2}\right)\right.\\
&\qquad\left.+\,2\Sigma(-1)^p\left(\frac{a_{1'}a_{1''}a_r}{\gamma_{1'1''}\gamma_{1'r}\gamma_{1''r}}+\frac{a_1a_{1''}a_r}{\gamma_{11''}\gamma_{1r}\gamma_{1''r}}+\frac{a_1a_{1'}a_r}{\gamma_{11'}\gamma_{1r}\gamma_{1'r}}\right)\times\frac{1}{\nu_1^2(\nu_1^2-\nu_r^2)}\right\}\\
&\qquad+\frac{2}{\nu_1^6}\,\frac{a_1a_{1'}a_{1''}}{\gamma_{1'1''}\gamma_{11''}\gamma_{11'}}
\end{aligned}\right\}=0. \tag{21}$$

As a verification of this equation, suppose each molecule to consist of one atom only, then the sum terms of course disappear and the equation agrees with that for the mutual influence of three equal atoms given on p. 86 (γ) of my first paper.

Suppose, on the other hand, that the molecules are so far apart that the vibrations of any atom depend only on the atoms of its own group. Then our cubic equation ought, by our Art. 9, equation (14), to reduce to

$$\left(\mu-\sum_1\frac{a_1a_s}{\gamma_{1s}^2}\,\frac{1}{\nu_1^2-\nu_s^2}\right)\left(\mu-\sum_2\frac{a_{1'}a_{s'}}{\gamma_{1's'}^2}\,\frac{1}{\nu_{1'}^2-\nu_{s'}^2}\right)\left(\mu-\sum_3\frac{a_{1''}a_{s''}}{\gamma_{1''s''}^2}\,\frac{1}{\nu_{1''}^2-\nu_{s''}^2}\right)=0. \tag{22}$$

Now (21) agrees with this if we neglect all terms depending on *molecular* distances, so far as the first two terms are concerned, i. e. we have

$$\mu^3-\mu^2\left\{\sum_1\frac{a_1a_s}{\gamma_{1s}^2}\,\frac{1}{\nu_1^2-\nu_s^2}+\sum_2\frac{a_{1'}a_{s'}}{\gamma_{1's'}^2}\,\frac{1}{\nu_{1'}^2-\nu_{s'}^2}+\sum_3\frac{a_{1''}a_{s''}}{\gamma_{1''s''}^2}\,\frac{1}{\nu_{1''}^2-\nu_{s''}^2}\right\}=0.$$

Nor should we expect it to go further because we have neglected all powers of a/γ above the third, or above the third and its product into μ, whereas the full expansion of (22) contains the sixth power of a/γ. Now it would be troublesome to expand the determinant (14) up to sixth powers of a_{rs}. But (21) will readily enable us to ascertain what effect the presence of the other two molecules has

on the vibrations of the first. This arises from the fact that equation (21) is a true approximation to the value of μ up to third powers of *inter-molecular* (as distinguished from *inter-atomic*) distances, and thus we have only to take equation (22) for μ and vary its constants by adding the inter-molecular terms from (21). Suppose that (22) can be written

$$\mu^3 - \lambda_1\mu^2 + \lambda_2\mu - \lambda_3 = 0. \tag{23}$$

Then the variations in this cubic equation due to inter-molecular action are expressed as follows:

$$\left.\begin{aligned}
&\mu^3 - \mu^2\left\{\lambda_1 + \sum_{2,3}\frac{a_1a_r}{\gamma_{1r}^2} + \frac{1}{\nu_1^2-\nu_r^2} + \sum_{1,3}\frac{a_{1'}a_r}{\gamma_{1'r}^2}\,\frac{1}{\nu_{1'}^2-\nu_r^2}\right.\\
&\qquad\qquad\left. + \sum_{2,1}\frac{a_{1''}a_r}{\gamma_{1''r}^2}\,\frac{1}{\nu_{1''}^2-\nu_r^2}\right\}\\
&+\mu\left\{\lambda_2 - \frac{1}{\nu_1^4}\left(\frac{a_{1'}a_{1''}}{\gamma_{1'1''}^2} + \frac{a_1a_{1''}}{\gamma_{11''}^2} + \frac{a_1a_{1'}}{\gamma_{11'}^2}\right)\right.\\
&\qquad - 2\sum_{1,2,3}(-1)^p\left(\frac{a_{1'}a_{1''}a_r}{\gamma_{1'1''}\gamma_{1'r}\gamma_{1''r}} + \frac{a_1a_{1''}a_r}{\gamma_{11''}\gamma_{1r}\gamma_{1''r}} + \frac{a_1a_{1'}a_r}{\gamma_{11'}\gamma_{1r}\gamma_{1'r}}\right)\\
&\qquad\qquad\left.\times\frac{1}{\nu_1^2(\nu_1^2-\nu_r^2)}\right\}\\
&-\left\{\lambda_3 - \frac{2}{\nu_1^6}\,\frac{a_1a_{1'}a_{1''}}{\gamma_{1'1''}\gamma_{11''}\gamma_{11'}}\right\}
\end{aligned}\right\} = 0. \tag{24}$$

Here r represents any atom of the three molecules, except the three $1, 1', 1''$ or which μ gives the period, and $\sum_{2,3}$ means a summation with regard to r of all atoms in both the 2^d and 3^d molecules except $1'$ and $1''$; $\sum_{1,2,3}$ means a summation with regard to r for all atoms in the 1^st, 2^d and 3^d molecules except $1, 1'$ and $1''$, and so on.

We shall denote the variations in the value of the coefficients of equation (23) as given in equation (24) by $\delta\lambda_1$, $\delta\lambda_2$ and $\delta\lambda_3$ respectively, thus we have to find μ from the equation

$$\mu^3 - \mu^2(\lambda_1 + \delta\lambda_1) + \mu(\lambda_2 + \delta\lambda_2) - (\lambda_3 + \delta\lambda_3) = 0,$$

or

$$\mu^3 - \mu^2\lambda_1 + \mu\lambda_2 - \lambda_3 - \mu^2\delta\lambda_1 + \mu\delta\lambda_2 - \delta\lambda_3 = 0.$$

Now for the mean sets of molecules the roots of equation (23) must all be equal, for they give the vibration of the one atom of three different molecules supposed to

be uninfluenced by each other, but in the same physical condition, i. e. the mean value of γ_{rs} = mean value of $\gamma_{r's'}$ = mean value of $\gamma_{r''s''}$, where γ_{rs} is the distance between r^{th} and s^{th} atom in the first molecule and $\gamma_{r's'}$, $\gamma_{r''s''}$ are like quantities for the other two molecules. Hence equation (22) has three equal roots $= \mu_1$, say. Thus the last equation may be written

$$(\mu - \mu_1)^3 - \mu^2 \delta\lambda_1 + \mu \delta\lambda_2 - \delta\lambda_3 = 0.$$

Let μ_1 be changed owing to the changes in the coefficients to $\mu_1 + \delta\mu_1$. Then we find

$$(\delta\mu_1)^3 - \delta\mu_1^2 \delta\lambda_1 - \delta\mu_1 (2\mu_1 \delta\lambda_1 - \delta\lambda_2) - \mu_1^2 \delta\lambda_1 + \mu_1 \delta\lambda_2 - \delta\lambda_3 = 0. \tag{25}$$

This is a cubic equation to find the three values of $\delta\mu_1$. Let us consider the order of the various terms of this equation, and suppose that to represent a quantity by $[1/a]^i [1/m]^j$ denotes that it is of the i^{th} order in the inverse of inter-atomic distance (i. e. distance between atoms of the same molecule), and that it is of the j^{th} order in the inverse of inter-molecular distance (i. e. in the distance between atoms of different molecules). Thus we find—

that μ is represented by			$[1/a]^2$,
" $\delta\lambda_1$	"	"	$[1/m]^2$,
" $\delta\lambda_3$	"	"	$[1/m]^3$,
" $\delta\lambda_2$	"	"	$[1/m]^3 + [1/m]^3 + [1/m]^2[1/a]$.

Hence we see that as a *first approximation* we may neglect $(\delta\mu_1)^2 \delta\lambda_1$ and $2\mu_1 \delta\mu_1 \delta\lambda_1$ as compared with $\delta\mu_1 \delta\lambda_2$; and further, $\mu_1^2 \delta\lambda_1$ as compared with the terms in $\mu_1 \delta\lambda_2$ of the order $[1/a]^3 [1/m]^2$. Whether we can neglect $\delta\lambda_3$ as compared with $\mu_1 \delta\lambda_2$ will depend on whether terms of order $[1/m]$ are comparable or not with terms of the order $[1/a]^2$. If the inverse of inter-molecular distance is only of the second order of small quantities as compared with the inverse of inter-atomic distance, then $\delta\lambda_3$ will have to be retained.

Thus we have to a first approximation for $\delta\mu$ the equation

$$\left.\begin{aligned} (\delta\mu_1)^3 - \frac{\delta\mu_1}{\nu_1^4}\left(\frac{a_{1'}a_{1''}}{\gamma_{1'1''}^2} + \frac{a_1 a_{1''}}{\gamma_{11''}^2} + \frac{a_1 a_{1'}}{\gamma_{11'}^2}\right) + \frac{2}{\nu_1^6}\,\frac{a_1 a_{1'} a_{1''}}{\gamma_{1'1''}\gamma_{11''}\gamma_{11'}} \\ - \frac{\mu_1}{\nu_1^4}\left(\frac{a_{1'}a_{1''}}{\gamma_{1'1''}^2} + \frac{a_1 a_{1''}}{\gamma_{11''}^2} + \frac{a_1 a_{1'}}{\gamma_{11'}^2}\right) \end{aligned}\right\} = 0. \tag{26}$$

14). From this equation we can draw some important conclusions:

a). The modified period of an atom in one molecule owing to the action

of other molecules in its neighborhood, is to a first approximation a function only of the relative positions of the *corresponding* atoms in the modifying molecules. Thus the vibrations of the atom A will depend not only on its distances from its kindred atoms A' and A'' in other molecules, but also on the distance of A' from A''; it will not, however (to a first approximation), depend upon A's distance from B' or C'', etc.

b). Suppose inter-molecular distance is enormously greater than inter-atomic distance, or $[1/m]$ very small as compared even with $[1/a]^2$. Then we neglect the first term in the constant of the equation for $\delta\mu_1$ as compared with the second. This shows us that $\delta\mu_1$ will be of the order $[1/a]^{\frac{4}{3}}[1/m]^{\frac{2}{3}}$, hence since $[1/m]$ is negligible as compared with $[1/a]^2$, it follows that the term involving $\delta\mu_1$ may be neglected as compared with $(\delta\mu_1)^3$, or we have

$$\delta\mu_1 = \left\{\frac{\mu_1}{\nu_1^4}\left(\frac{a_{1'}a_{1''}}{\gamma_{1'1''}^2} + \frac{a_1 a_{1''}}{\gamma_{11''}^2} + \frac{a_1 a_{1'}}{\gamma_{11'}^2}\right)\right\}^{\frac{1}{3}}. \tag{27}$$

In this case there is only *one real* value of $\delta\mu_1$, the same for all three kin-atoms of the three molecules. Thus we should expect, if the molecules be so far apart that they have no modifying action and so exhibit a pure bright line spectrum, that when they are brought closer together there would first be a slight shifting of their bright lines, and this shifting—since $\delta\mu_1$ is positive, denoting a decrease in the period—is towards the violet end of the spectrum. So soon, however, as the effect of pressure or temperature causes $[1/m]$ to be comparable with $[1/a]^2$, the individual bright line is replaced by three others, if only two molecules have modifying influence on a third, but by $(p+1)$ bright lines if p molecules exercise such influence. Here we see, perhaps, a little more clearly than in the first paper, how the bright line spectrum becomes fluted and ultimately continuous.

In this case the variation in μ_1 due to the modifying action of the kindred atoms in other molecules, is of the order $[1/a]^{\frac{2}{3}}[1/m]^{\frac{2}{3}}$, or its ratio to μ_1 of the order $(a/m)^{\frac{2}{3}}$. That is, when inter-molecular distance is very great as compared with inter-atomic, the modification in period of any atom is of the order

$$\left(\frac{\text{intera-tomic distance}}{\text{inter-molecular distance}}\right)^{\frac{2}{3}}.$$

c). Suppose $[1/m]$ of the order $[1/a]^2$. Then we must retain all terms of the equation (26). We see that $\delta\mu_1$ will be of the order $[1/a]^2$, but this is pre-

cisely the same order as μ_1 itself, or we conclude: That modified action becomes as important as inter-atomic action in settling the period of an atom if the inverse of inter-molecular distance is comparable with the inverse square of inter-atomic distance. The reason of this apparent anomaly lies in the fact that an atom will produce greater effect on a kindred atom when at a considerably greater distance than on non-kindred atoms at a considerably less distance. In deducing this result we have supposed that there are *not* two equal or kindred atoms in any one molecule. If there were, the quantity μ_1 is not of the order $[1/a]^2$, but of the order $[1/a]$, and is therefore sensibly greater than the modifying effect of kindred atoms in other molecules.

d). If $[1/m]$ be sensibly greater than $[1/a]^2$, then the equation for $\delta\mu_1$ reduces to that for the mutual influence of three equal atoms,* and the variation in μ_1 is simply got by adding to the value of μ_1 obtained from the isolated molecule, the value of μ_1 which would be obtained from treating the three kindred atoms as isolated in space.

With regard to all the above cases we must remark that inter-molecular action is accumulative, i. e. a great number of molecules may be near enough to affect the period of vibration of an atom of the molecule under consideration, and we may have to sum a very great number of small terms, so increasing the total effect.

According to Ampère and Becquerel, inter-molecular distance is enormous as compared with molecular diameter; according to Babinet, it is at least as 1800:1. Sir W. Thomson, however, considers that the mean distance of two contiguous molecules of a solid is less than $\frac{1}{100,000,000}$ of a centimeter, while the diameter of a *gaseous* molecule is greater than $\frac{1}{500,000,000}$ of a centimeter. Supposing the diameter of a molecule not to differ very much in the solid and gaseous conditions, this would lead to inter-molecular distance being less than five times inter-atomic distance, or to the influence of molecules on the atomic vibrations of each other being very considerable.† Although it is not certain that the ratio is so small as this, still the phenomena of continuous and fluted spectra, as well as the possibility of multi-constant elasticity, lead us

* See p. 86 (γ) of my first paper.

† See Thomson and Tait: *Natural Philosophy*, Part II, p. 502, and Todhunter and Pearson: *History of Elasticity*, Vol. II, p. 184.

to believe that inter-molecular action on atomic vibrations is very sensible, and therefore to adopt what Jellett has termed the *Hypothesis of modified Action.*

e). It may not be without interest to mark the exact effect of non-kindred atoms in other molecules on the atomic vibrations of a given molecule.

Let μ' be a root of equation (24) when we neglect the modifying action of non-kindred atoms, so that $\mu_1' = \mu_1 + \delta\mu_1$. Thus we may write

$$\mu_1'^3 - \mu_1'^2\lambda_1' + \mu_1'\lambda_2' - \lambda_3' = 0,$$

where the dashed λ's are the λ's of equation (23) affected only by the modifying action of kindred atoms. Now, owing to the action of the non-kindred atoms, λ_1' becomes $\lambda_1' + \delta\lambda_1'$ and λ_2' becomes $\lambda_2' + \delta\lambda_2'$. The effect on μ_1', which we will represent by $\delta\mu_1'$, is easily found to be

$$\delta\mu_1' = \frac{\mu_1'(\mu_1'\delta\lambda_1' - \delta\lambda_2')}{3\mu_1'^2 - 2\mu_1'\lambda_1' + \lambda_2'}.$$

The denominator $= 3(\mu_1 + \delta\mu_1)^2 - 2(\mu_1 + \delta\mu_1)(\lambda_1 + \delta\lambda_1) + \lambda_2 + \delta\lambda_2$, but $3\mu_1^2 - 2\mu_1\lambda_1 + \lambda_2 = 0$ and $\lambda_1 = 3\mu_1$. Hence we have the denominator $= 3(\delta\mu_1)^2 - 2\mu_1\delta\lambda_1 - 2\delta\mu_1\delta\lambda_1 + \delta\lambda_2$. Considering the order of terms, we see that the two mid-terms are small as compared with the extreme terms, and thus we may take the denominator $= 3(\delta\mu_1)^2 + \delta\lambda_2$. Turning to the numerator, we have

$\mu_1'\delta\lambda_1'$ represented by $[1/a]^2[1/m]^2$

and $\delta\lambda_2'$ " " $[1/m]^3 + [1/m]^2[1/a]$.

Thus the second part of the latter is the most important, or we may write

$$\delta\mu_1' = \frac{-\mu_1'\delta\lambda_2'}{3(\delta\mu_1)^2 + \delta\lambda_2}. \tag{28}$$

We ought to distinguish two cases—

(i). $[1/m]$ is immensely greater than $[1/a]$ and not comparable even with $[1/a]^2$.

Hence $\delta\lambda_2$ may be neglected as compared with $\delta\mu_1$ which is now given by equation (27); further, μ_1' may be replaced by μ_1 and

$$\left.\begin{aligned}\delta\lambda_2' = -\frac{2}{\nu_1^2}\Bigg\{ &\sum_{23}(-1)^p \frac{a_{1'}a_{1''}a_r}{\gamma_{1'1''}\gamma_{1'r}\gamma_{1''r}}\,\frac{1}{(\nu_1^2 - \nu_r^2)} \\ &+ \sum_{1,3}(-1)^p \frac{a_1 a_{1''}a_r}{\gamma_{11''}\gamma_{1r}\gamma_{1''r}}\,\frac{1}{(\nu_1^2 - \nu_r^2)} \\ &+ \sum_{1,2}(-1)^p \frac{a_1 a_{1'}a_r}{\gamma_{11'}\gamma_{1r}\gamma_{1'r}}\,\frac{1}{(\nu_1^2 - \nu_r^2)}\Bigg\}\end{aligned}\right\}. \tag{29}$$

Thus we see that the order of $\delta\mu_1'$ is represented by

$$\frac{[1/a]^3[1/m]^2}{[1/a]^{\frac{4}{3}}[1/m]^{\frac{4}{3}}} = [1/a]^{\frac{5}{3}}[1/m]^{\frac{2}{3}}.$$

Thus the ratio of $\delta\mu_1'$ to $\delta\mu_1$ is of the order $[1/a]^2$, or if $\delta\mu_1$ be of the second order of small quantities, this modification is of the *fourth order*.

The character of the disturbance in μ_1 as given by $\delta\lambda_2'$ is noteworthy. It does not involve the inter-molecular distances of kindred atoms other than those the variation of the period of which we are dealing with. Thus if A, B, C, ... A', B', C', ... A'', B'', C'', ... denote the atoms of the three molecules, A, A', A'' being kindred atoms, then the manner in which B', B'', C', C'' modify the period of A is by their distances from A, A' and A'' and not by their distances from B or C, or even from B' and C'; the latter are distances which it would seem probable might have occurred.

Case (ii). $[1/m]$ is of the same order as $[1/a]^2$.

Here the term of the order $[1/m]^2$ in $\delta\lambda_2$ must be retained in the denominator and $\mu_1 + \delta\mu_1$ must be put for μ_1' in the numerator. The order of $\delta\mu_1'$ will now be represented by $[1/a]^3$, or the ratio of $\delta\mu_1'$ to $\delta\mu_1$, or to μ_1, is only of the order $[1/a]$, thus its importance has much increased owing to the reduction in molecular distance. The exact value of $\delta\mu_1'$ may be easily written down from equations (28) and (29).

The general conclusion we seem compelled to form is, that the period of atomic vibrations will be sensibly modified by inter-molecular action, so soon as $[1/m]$ is not negligible as compared with $[1/a]^2$.

15). We pass now from the consideration of the modifying action on period to the modifying action on amplitude. Let C_s denote the amplitude of the term of period $2\pi/n$ in the s^{th} atom. It will be sufficient to determine all these amplitudes if we investigate types of each period. We shall endeavor, then, to find the relation between C_1, C_2, $C_{1'}$ and $C_{2'}$, when $n =$ one of the values found above for $\mu_1 + \delta\mu_1 + \delta\mu_1'$.

Let $\alpha_{rs} =$ the minor of the determinant obtained by the r^{th} row and s^{th} column of the form in equation (18); then by the usual theory for the solution of equations of the type (11), we have

$$\frac{C_1}{\alpha_{11}} = \frac{C_{1'}}{\alpha_{11'}} = \frac{C_2}{\alpha_{12}} = \frac{C_{2'}}{\alpha_{12'}}.$$

We have now to find approximate values for these minors. Let us take them in order.

We have $a_{11} = D_1$ of our Art. 12, but it will not do to take for its value simply the expression there cited in terms of the elements, because that expression only gives it so far as the cubes of small elements, our object being then merely to find how the terms of a known cubic were affected by inter-molecular distances up to the cubes. We have now to calculate out D_1 as far as the fourth powers of inter-atomic and the square powers of inter-molecular distance. Now

$$D_1 = a_{11} = \begin{vmatrix} T_{1'} & a_{1'1''} & a_{1'2} & a_{1'2'} & a_{1'2''} \dots \\ a_{1''1'} & T_{1''} & a_{1''2} & a_{1''2'} & a_{1''2''} \dots \\ a_{21'} & a_{21''} & T_2 & a_{22'} & a_{22''} \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix},$$

where $T_r = \tau_r \left(\frac{1}{\nu_r^2} - \frac{1}{n^2}\right)$ and $a_{rs} = \frac{1}{\gamma_{rs}}$. Here $T_{1'}$ and $T_{1''}$ are of the order $[1/a]^2$, but T_r for all values except $1'$ and $1''$ is supposed not small. Let us pick out all terms of order equal or less than $[1/a]^4$. If we retain all the large T's we have the terms $T_2 \dots T_{k''} (T_{1'}T_{1''} - a_{1'1''}^2)$. Let us represent the product of all large T's by P, the above term is thus

$$P(T_{1'}T_{1''} - a_{1'1''}^2).$$

Now omit one large T, say T_r, and select (i) an inter-molecular term, say $a_{r'r}$, out of the r^{th} column; this will have to be multiplied by $a_{r1'}$ and $T_{1''}$, it is thus of the order $[1/m]^2[1/a]^2$ and to be neglected; (ii) an inter-atomic term; this is impossible. Hence T_r gives us no terms. Now consider $T_{r'}$; we may now take the inter-atomic term $a_{1'r'}$ and it must be multiplied by $a_{r'1'}T_{1''}$. Thus we have two expressions of the form

$$T_{1''}P \sum_2 \frac{a_{1'r'}^2}{T_{r'}} \text{ and } T_{1'}P \sum_3 \frac{a_{1''r''}^2}{T_{r''}},$$

and a little consideration shows that the signs of these terms are *negative*. They are all we get to the order $[1/a]^4$ by omitting one large T.

Now let us omit two large T's. As a first case say T_r and T_s corresponding to atoms both in the first molecule, then the quantities to be chosen out of the 1^{st}, 2^{d}, r^{th} and s^{th} columns must all be inter-atomic, but this is easily seen to be

impossible. As a second case take T_r and $T_{s'}$. We have to take two elements out of the first two rows; they cannot be taken out of the first two columns because these are of the order $[1/a]^2$ at least, therefore we must choose $a_{1'r'}$, but this throws us back on $a_{1''s'}$, which is inter-molecular. Hence no terms such as $T_{r'}$ and $T_{s'}$ can be omitted. Now let us omit $T_{r'}$ and $T_{s''}$, we are then compelled to take the inter-atomic terms $a_{r'1'}^2 a_{s''2''}^2$. The sign of these terms is found to be positive, or we have the expression

$$P\Sigma \frac{a_{r'1'}^2 a_{s''2''}^2}{T_{r'} T_{s''}}.$$

As a fourth case let us take out two terms like T_r and $T_{s'}$; then the choice of an inter-atomic term out of the r^{th} column becomes impossible, and thus this form of term does not appear to our degree of approximation.

Finally we have

$$\begin{aligned} D_1 &= P\left\{ T_1 T_{1''} - T_{1''} \sum_2 \frac{a_{1'r'}^2}{T_{r'}} - T_{1'} \sum_3 \frac{a_{1''r''}^2}{T_{r''}} - a_{1'1''}^2 + \Sigma \frac{a_{r'1'}^2 a_{s''2''}^2}{T_{r'} T_{s''}} \right\} \\ &= P\left\{ \left(T_{1'} - \sum_3 \frac{a_{1''r''}^2}{T_{r''}} \right)\left(T_{1''} - \sum_2 \frac{a_{1'r'}^2}{T_{r'}} \right) - a_{1'1''}^2 \right\}. \end{aligned}$$

Thus
$$\alpha_{11} = P\left\{ \left(\tau_{1'}\mu - \sum_3 \frac{a_{r''}}{\gamma_{1''r''}^2} \frac{n^2}{n^2 - \nu_r^2} \right)\left(\tau_{1''}\mu - \sum_2 \frac{a_{r'}}{\gamma_{1'r'}^2} \frac{n^2}{n^2 - \nu_{r'}^2} \right) - \frac{1}{\gamma_{1'1''}^2} \right\}.$$

Now
$$\sum_3 \frac{a_{r''}}{\gamma_{1''r''}^2} \frac{n^2}{n^2 - \nu_r^2} = \sum_3 \frac{a_{r''}}{\gamma_{1''r''}^2} \frac{\nu_1^2}{\nu_1^2 - \nu_r^2} = \tau_{1'}\mu_1,$$

if we neglect in the value of α_{11} terms of the order $(a/\gamma)^6$. Now $\mu = \mu_1 + \delta\mu_1$, hence

$$\alpha_{11} = P\tau_{1'}\tau_{1''}\left\{ (\delta\mu_1)^2 - \frac{a_{1'}a_{1''}}{\nu_1^4 \gamma_{1'1''}^2} \right\}. \tag{30}$$

Let us now find $\alpha_{1'1}$ to the same order of small quantities. In precisely the same manner we have

$$\left.\begin{aligned} \alpha_{1'1} &= -P\left\{ a_{1'1} T_{1''} - \sum_3 \frac{a_{1''r''}^2 a_{1'1}}{T_{r''}} - a_{1'1''}^2 \right\} \\ &= -P\left\{ \frac{\tau_{1''}(\mu - \mu_1)}{\gamma_{1'1}} - \frac{1}{\gamma_{1'1''}^2} \right\} \\ &= -\frac{P\tau_{1'}\tau_{1''}}{\nu_1^2}\left\{ \frac{a_{1'}}{\gamma_{1'1}}\delta\mu_1 - \frac{1}{\nu_1^2} \frac{a_{1'}a_{1''}}{\gamma_{1'1''}^2} \right\} \end{aligned}\right\}. \tag{31}$$

Further we find

$$\left.\begin{aligned} \alpha_{12} &= -\frac{P\tau_{1'}\tau_{1''}}{\gamma_{12} T_2}\left\{ (\delta\mu_1)^2 - \frac{a_{1'}a_{1''}}{\nu_1^4 \gamma_{1'1''}^2} \right\} \\ &= -P\tau_{1'}\tau_{1''} \frac{\nu_1^2}{\nu_1^2 - \nu_2^2} \frac{a_2}{\gamma_{12}}\left\{ (\delta\mu_1)^2 - \frac{a_{1'}a_{1''}}{\nu_1^4 \gamma_{1'1''}^2} \right\} \end{aligned}\right\}. \tag{32}$$

Finally, we have

$$\alpha_{12'} = P\frac{\tau_{1'}\tau_{1''}}{\nu_1^2-\nu_{2'}^2}\frac{a_{2'}}{\gamma_{1'2'}}\left\{\frac{a_{1'}}{\gamma_{11'}}\delta\mu_1-\frac{a_{1'}a_{1''}}{\nu_1^2\gamma_{1'1''}^2}\right\}. \tag{33}$$

We can now write down the values of $C_{1'}$, C_2, $C_{2'}$ in terms of C_1, we find

$$C_2 = -\frac{\nu_1^2}{\nu_1^2-\nu_2^2}\frac{a_2}{\gamma_{12}}C_1. \tag{34}$$

Suppose $[1/m]$ not comparable with $[1/a]^2$, then

$$C_{1'} = -\frac{1}{\nu_1^2}\frac{a_{1'}}{\gamma_{11'}}\frac{1}{\delta\mu_1}C_1, \tag{35}$$

$$C_{2'} = \frac{1}{\nu_1^2-\nu_{2'}^2}\frac{a_{2'}a_{1'}}{\gamma_{1'2'}\gamma_{11'}}\frac{1}{\delta\mu_1}C_1. \tag{36}$$

These results, although only approximations, seem to be of very considerable interest. I proceed to note some points connected with them.

a). Equation (34) is precisely the same equation as (15), determining the relation between the amplitudes of the 1st and 2d atoms of the same molecule, when no other molecule is affecting their vibrations. Thus it would seem that, to the approximation we have adopted, the 1st atom produces the same amplitudes in the atoms of its own molecule notwithstanding the presence of other molecules in the field. All we can safely say, however, is that the all-important or leading term in the amplitude remains unchanged. Other small terms may be added to it, for although we have calculated α_{11} and α_{12} to the order $[1/a]^4$, we have divided them both by a factor of the order $[1/a]^4$ when finding the ratio of C_1 to C_2. Hence to have obtained the modifying influence of the other molecules we ought to have calculated the minors to $[1/a]^6$. At the same time we shall see immediately that the influence of C_2 on $C_{2'}$ is exerted indirectly, i. e. by means of its influence on $C_{1'}$; hence it seems more than probable that the modifying influence of other molecules on C_2 (the amplitude corresponding to C_1) is exerted by altering C_1, or also indirectly.

b). Equation (35) shows us that when $[1/m]$ is not comparable with $[1/a]^2$, then since $\delta\mu_1$ is small, there is a very sensible amplitude $C_{1'}$ produced in the first atom of the second molecule by the action of the kindred atom in the first. By equation (27) $(\delta\mu_1)$ is of the order $[1/m]^{\frac{1}{2}}$. Hence $C_{1'}/C_1$ is of the order $[1/m]^{\frac{1}{2}}$, or *if* $[1/m]$ *be comparable only with* $[1/a]^2$, *still the effect of the first atom in the first molecule on its kindred atom in the second, will be as great as its effect on the other atoms of its own molecule, i. e. both will be of the order* $[1/a]$.

c). By equation (34) as type we should have

$$C_{2'} = -\frac{\nu_{1'}^2}{\nu_{1'}^2 - \nu_{2'}^2}\,\frac{a_{2'}}{\gamma_{1'2'}}\,C_{1'}.$$

This gives the effect of an amplitude $C_{1'}$ in the $1'^{\text{th}}$ atom on the amplitude of the $2'^{\text{d}}$ atom; or, if $C_{1'}$ be the variation in amplitude of the $1'^{\text{th}}$ atom due to the 1^{st} atom of the first molecule, we have by (35):

$$C_{2'} = -\frac{\nu_{1'}^2}{\nu_{1'}^2 - \nu_{2'}^2}\,\frac{a_{2'}}{\gamma_{1'2'}}\left(-\frac{1}{\nu_1^2}\,\frac{a_{1'}}{\gamma_{11'}}\,\frac{1}{\delta\mu_1}\,C_1\right) = \frac{1}{\nu_1^2 - \nu_{2'}^2}\,\frac{a_{2'}a_{1'}}{\gamma_{1'2'}\gamma_{11'}}\,\frac{1}{\delta\mu_1}\,C_1,$$

which is exactly the value given by (36). This proves the point I have referred to above, namely, that whatever influence non-kindred atoms in different molecules exert on each other is indirect and takes place through their effect on their kindred atoms. This of course is only true for the approximation to which we have gone. The effect is, indeed, of a higher order of small quantities than that of the 1^{st} atom on the 2^{d} or on the $1'^{\text{th}}$. If $[1/m]$ is comparable with $[1/a]^3$ it will be of the $[1/a]^2$ order, while the latter are of the $[1/a]$ order.

We may conclude, then, that when molecular distance is such that $[1/m]$ cannot be compared with $[1/a]^2$, then the modifying action of two molecules on each other's atomic vibrations is practically limited to the mutual action of their kindred atoms.

Now let us suppose $[1/m]$ is comparable with $[1/a]$. What we have said above about equation (34) still holds, but we find for $C_{1'}$ and $C_{2'}$

$$C_{1'} = -\frac{\dfrac{a_{1'}}{\gamma_{11'}}\,\dfrac{\delta\mu_1}{\nu_1^2} - \dfrac{1}{\nu_1^4}\,\dfrac{a_{1'}a_{1''}}{\gamma_{1'1''}^2}}{(\delta\mu_1)^2 - \dfrac{1}{\nu_1^4}\,\dfrac{a_{1'}a_{1''}}{\gamma_{1'1''}^2}}\,C_1, \tag{37}$$

$$C_{2'} = \frac{\nu_1^2}{\nu_1^2 - \nu_{2'}^2}\,\frac{a_{2'}}{\gamma_{1'2'}}\,\frac{\dfrac{a_{1'}}{\gamma_{11'}}\,\dfrac{\delta\mu_1}{\nu_1^2} - \dfrac{1}{\nu_1^4}\,\dfrac{a_{1'}a_{1''}}{\gamma_{1'1''}^2}}{(\delta\mu_1)^2 - \dfrac{1}{\nu_1^4}\,\dfrac{a_{1'}a_{1''}}{\gamma_{1'1''}^2}}\,C_1. \tag{38}$$

Now the numerator and denominator on the right-hand side of equation (37) are both of the same order, or we draw the conclusion that $C_{1'}$ and C_1 are of the same order. This follows from the fact that $\delta\mu_1$ is of the order $[1/a]^2$: see our

Art. 14 (c). Further, the effect of C_1 on $C_{2'}$ is as before indirect and due to the effect of C_1 on $C_{1'}$, but its order as compared with C_1 is now $[1/a]$. We are thus able to draw the following important conclusions for the case of $[1/m]$ being comparable with $[1/a]^2$:

a$'$). The variation in the amplitude of an atom due to the vibration of a kindred atom in another molecule is of the same order as the amplitude of the second atom.

b$'$). The variation in the amplitude of one atom due to the vibration of a non-kindred atom in another molecule is indirect, being produced by the influence exerted by the second atom on its kindred atom in the first molecule. It is of the same importance, however, as the variations produced in the amplitude of the same atom by any atom in its own molecule.

c$'$). The variations produced in the amplitudes of the atoms of one molecule by an atom of another are sensibly affected by the presence of a third molecule. This effect depends, in the approximation to which we have gone, entirely on the distance between those atoms of the first and third molecules which are kindred to the modifying atom of the second.

It should be noticed that owing to the value of $\delta\mu_1$, a remark similar to the last, (c$'$), may be made for the system (35) and (36), where $\delta\mu_1$ is determined by (27). The modifying action of the presence of the third molecule will not be so perceptible, however, as the variations in amplitude are so much less.

From the above statements flow the truth of the *hypothesis of modified action* and of the multi-constant equations of elasticity for matter built up of ether squirts. This will appear still more clearly if we determine the general form* of the inter-molecular force. To this we devote the following article.

* I ought to note here that the above equations are susceptible of almost infinite variety. Their complexity becomes very considerable when, instead of only one atom of each kind in each molecule, we place p equal atoms in each k-atomic molecule—this being probably a very ordinary occurrence in nature. Further interesting results arise when we have two different atoms, whose periods are, however, nearly equal. In this case we have more diagonal terms of the fundamental determinant *small*, and the approximations must be carried still further. In fact it may be safely said that every substance, the atomic structure of the molecules of which is known, would require a mathematical dissertation to itself. Nor ought this complexity to discourage us. Whatever be the element of matter, we expect it to possess great simplicity of structure (and this is at least satisfied in the ether squirt), but this very simplicity has to explain the nigh infinite range of chemical and physical properties in each individual substance, or it must lead us to a mathematical theory of the substance of the very highest complexity in itself.

16). From the preceding article we find the following values for ϕ_1, $\phi_{1'}$ and ϕ_2 as types of vibration:

$$\left.\begin{aligned}\phi_1 = {} & C_1 \cos(n_1 t + \alpha_1) + x_{1'} C_{1'} \cos(n_{1'} t + \alpha_{1'}) + x_{1''} C_{1''} \cos(n_{1''} t + \alpha_{1''}) \\ & - \sum^{1} \frac{\nu_r^2}{\nu_r^2 - \nu_1^2} \frac{a_1}{\gamma_{1r}} C_r \cos(n_r t + \alpha_r) \\ & + \Sigma_1^2 \lambda_{1,r'} C_{r'} \cos(n_{r'} t + \alpha_{r'}) \\ & + \Sigma_1^3 \lambda_{1,r''} C_{r''} \cos(n_{r''} t + \alpha_{r''}) \,.\end{aligned}\right\}, \qquad (39)$$

$$\left.\begin{aligned}\phi_{1'} = {} & x'_1 C_1 \cos(n_1 t + \alpha_1) + C_{1'} \cos(n_{1'} t + \alpha_{1'}) + x'_{1''} C_{1''} \cos(n_{1''} t + \alpha_{1''}) \\ & - \sum^{2} \frac{\nu_{r'}^2}{\nu_{r'}^2 - \nu_{1'}^2} \frac{a_{1'}}{\gamma_{1'r'}} C_{r'} \cos(n_{r'} t + \alpha_{r'}) \\ & + \Sigma_2^1 \lambda'_{1'r} C_r \cos(n_r t + \alpha_r) \\ & + \Sigma_2^3 \lambda'_{1'r''} C_{r''} \cos(n_{r''} t + \alpha_{r''})\end{aligned}\right\}, \qquad (40)$$

$$\left.\begin{aligned}\phi_2 = {} & C_2 \cos(n_2 t + \alpha_2) + x_{2'} C_{2'} \cos(n_{2'} t + \alpha_{2'}) + x_{2''} C_{2''} \cos(n_{2''} t + \alpha_{2''}) \\ & - \sum^{1} \frac{\nu_r^2}{\nu_r^2 - \nu_2^2} \frac{a_2}{\gamma_{2r}} C_r \cos(n_r t + \alpha_r) \\ & + \Sigma_1^2 \lambda_{2,r'} C_{r'} \cos(n_{r'} t + \alpha_{r'}) \\ & + \Sigma_1^3 \lambda_{2,r''} C_{r''} \cos(n_{r''} t + \alpha_{r''})\end{aligned}\right\}, \qquad (41)$$

where
$$x_{s'} = -\frac{\dfrac{a_s}{\gamma_{ss'}} \dfrac{\delta\mu_{s'}}{\nu_{s'}^2} - \dfrac{1}{\nu_{s'}^4} \cdot \dfrac{a_s a_{s''}}{\gamma_{ss''}^2}}{(\delta\mu_{s'})^2 - \dfrac{1}{\nu_{s'}^4} \dfrac{a_s a_{s''}}{\gamma_{ss''}^2}},$$

and $x_{s''}$ equals the value of $x_{s'}$ with the subscripts s' and s'' interchanged.

$$x'_s = -\frac{\dfrac{a_{s'}}{\gamma_{ss'}} \dfrac{\delta\mu_s}{\nu_s^2} - \dfrac{1}{\nu_s^4} \dfrac{a_{s'} a_{s''}}{\gamma_{s's''}^2}}{(\delta\mu_s)^2 - \dfrac{1}{\nu_s^4} \dfrac{a_{s'} a_{s''}}{\gamma_{s's''}^2}},$$

and $x'_{s''}$ equals the value of x'_s with the subscripts s and s'' interchanged.

$$\lambda_{s,r'} = \frac{\nu_{r'}^2}{\nu_{r'}^2 - \nu_s^2} \frac{a_s}{\gamma_{rs}} \frac{\dfrac{a_r}{\gamma_{rr'}} \dfrac{\delta\mu_{r'}}{\nu_{r'}^2} - \dfrac{1}{\nu_{r'}^4} \dfrac{a_r a_{r''}}{\gamma_{rr''}^2}}{(\delta\mu_{r'})^2 - \dfrac{1}{\nu_{r'}^4} \dfrac{a_r a_{r''}}{\gamma_{rr''}^2}},$$

and $\lambda_{s,r''}$ is the same expression with r' changed to r'' and r'' to r'.

$$\lambda'_{s'r} = \frac{\nu_r^2}{\nu_{r'}^2 - \nu_{s'}^2} \frac{a_{s'}}{\gamma_{r's'}} \frac{\frac{a_{r'}}{\gamma_{r'r}} \frac{\delta\mu_r}{\nu_r^2} - \frac{1}{\nu_r^4} \frac{a_{r'}a_{r''}}{\gamma_{r'r''}^2}}{(\delta\mu_r)^2 - \frac{1}{\nu_r^4} \frac{a_{r'}a_{r''}}{\gamma_{r'r''}^2}},$$

and $\lambda'_{s',r''}$ is the same expression with the subscripts r and r'' interchanged.

Finally $\overset{1}{\Sigma}$ denotes a summation for all the atoms of the first molecule except those giving an infinite value to the subject of summation. $\overset{2}{\Sigma}$ has a like value for the second molecule. Further, Σ_2^1 denotes a summation of all values of r for the first molecule, r' in each term being given the corresponding value from the second molecule, Σ_2^3, Σ_1^2, etc., have like meanings; in all cases those values of r, r' or r'' being excluded which give rise to infinite terms.

The above results agree with those of my first paper if we put the $\varkappa$'s and λ's zero, or neglect the semi-chemical action of kindred atoms in producing inter-molecular force (see §42, p. 105, of that paper).

Now, in order to determine inter-molecular force, we must find the value of such terms as

$$\frac{1}{4\pi\rho} \frac{\dot{\phi}_1 \dot{\phi}_{1'}}{\gamma_{11'}} \text{ and } \frac{1}{4\pi\rho} \frac{\dot{\phi}_1 \dot{\phi}_{2'}}{\gamma_{12'}},$$

which are types of those occurring in the force function of equation (9). This involves ascertaining the mean values of such expressions as $\dot{\phi}_1 \dot{\phi}_{1'}$ and $\dot{\phi}_1 \dot{\phi}_{2'}$, and this will depend on the relative magnitude of inter-molecular and inter-atomic distances. We shall treat accordingly the following cases:

(i). Inter-molecular distance is quite incomparable with inter-atomic distance (i. e. $[1/m]^{\frac{1}{3}}$ is negligible as compared with $[1/a]$.

(ii). Inter-molecular distance is very great as compared with inter-atomic distance, but $[1/m]^{\frac{1}{3}}$ cannot be neglected as compared with $[1/a]$.

(iii). Inter-molecular distance is comparable with $[1/a]^2$.

These cases may be represented by

$$\begin{aligned} &\text{(i)} \quad [1/m] < [1/a]^3, \\ &\text{(ii)} \quad [1/m] = [1/a]^3, \\ &\text{(iii)} \quad [1/m] = [1/a]^2. \end{aligned}$$

If $[1/m] = [1/a]$, we obviously have complete chemical union between molecules, and therefore molecules cease to exist as such. This case we may exclude from our consideration.

Now, an important point still remains unsettled, namely: Is there any relation between the phases of the like vibrational terms in two like molecules? This point is only important for cases (i) and (ii), but for them it is of crucial importance. If two like molecules be at such distances as in no way to affect each other's periods, the vibrations characteristic of kin-atoms will be of the same period: will they also be of the same phase? In my first paper I defined molecular force to arise from the interaction of the *free* vibrations of the molecules. I did not suppose the molecules to form a single system with mutually *forced* vibrations, or inter-molecular force partially dependent on a semi-chemical action between kin-atoms of different molecules. Further investigation of the fundamental determinant has taught me the importance of this action at least for the case $[1/m] = [1/a]^2$.

Now the results of this assumption or rather omission in my first paper were the following:

(i). Supposing the phases of the like vibrations in molecules of the same substance to be unequal, then the first term in inter-molecular action is an attractive force varying as the inverse *fifth power* of the distance (p. 106). The existence of this force depends on the particular atomic hypothesis of pulsating spheres of finite, if small, radii. On that hypothesis a term of the form $\frac{a_1^3 a_{1'}^3}{\gamma_{11'}^4}(a_1\dot{\phi}_1^2 + a_{1'}\dot{\phi}_{1'}^2)$ as well as the term of form $\frac{\dot{\phi}_1\dot{\phi}_{1'}}{\gamma_{11'}}$ arises in the force function of the molecules. The former term, however, disappears on the present hypothesis of ether squirts, which reduces the force function solely to terms of the order $\frac{\dot{\phi}_1\dot{\phi}_{1'}}{\gamma_{11'}}$. Hence we see that on the hypothesis of ether squirts there would be no *molecular force*, when we neglected the semi-chemical action between molecules, unless the corresponding vibrations of the molecules were in the same phase. Is there any reasoning by which we can prove this result invalid? At first sight, apparently not, especially in our ignorance of the relative magnitude of inter-atomic and inter-molecular distances. In a distended gas the semi-chemical action might be negligible and there would thus be no sensible cohesive force; there is no objection to this.

Such a gas could not be self-incandescent or send forth rays of light without the application of external energy, for this self-incandescence would involve, I think, the vibrations of a great number of molecules being in the *same phase*. On the other hand, it is possible that the means taken to produce the bright line spectrum of such a gas really sets a great number of molecules vibrating in the same phase.* The argument from gaseous bodies giving light, although weighty, is thus not conclusive. If we suppose two free molecules not to be vibrating in the same phase, we are thrown back on a semi-chemical action between the kin-atoms of different molecules as the basis of cohesion. Possibly the effects of altering the chemical constitution of a molecule by increasing the number of like atoms in it, which would affect the cohesion and so the volume of a substance, might tend to throw some light on this point.

(ii). If two free molecules have vibrations in equal phase, or if two kin-atoms have free vibrations of equal phase, and their forced vibrations differ only by terms of the order (atomic radius/atomic distance)^2, then cohesive force varies partly as the inverse square and partly as the inverse fifth power of molecular distance. But so far as these terms are concerned there is no modifying action due to the presence of other molecules in the field (see §§42 and 47 of my first paper).

There is something to be said for the possibility of all equal atoms having started vibrating in the same phase—for what might be termed an instant of 'creation.' The effect of inter-atomic action would be principally felt in a variation of period, and the slight variations in change of phase, if they have any existence, might be of the order (atomic radius/inter-atomic distance)^3 and thus the differences of phase might not amount to a quarter period. It is thus a possibility which we must keep in view in evaluating inter-molecular force, and I shall therefore deal with the above cases under the two headings of 'Equal Phase,' by which I shall mean absolutely equal, or only very small differences, and 'Unequal Phase,' by which I shall connote that the difference of phase may take all possible values.

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* Suppose a number of small tuning-forks at such distances that the feeble air vibrations they produce cannot materially affect each other, although the forks have the same note. They may then all be vibrating in *different* phases. Now a very large tuning-fork, caused to vibrate with this note, might set a whole group of the lesser ones, from which it was about equally distant, vibrating in the *same* phase, and therefore represent the manner in which the spectrum of a gas could arise.

17). Case (i). $[1/m] < [1/a]^3$. (Condition of rare gas?).
Here the expressions for $\dot\phi_1$ and $\dot\phi_{1'}$ reduce to those of my first paper (p. 105).

Sub-case (a). *Equal Phase.*

The mean values of the expressions $\dot\phi_s\dot\phi_{s'}$ and $\dot\phi_q\dot\phi_{s'}$ are now

$$\dot\phi_s\dot\phi_{s'} = \tfrac{1}{2}\, C_s C_{s'} n_s^2 + \overset{(r \text{ and } r' \text{ all values but } s \text{ and } s'.)}{\Sigma} \frac{\nu_r^4 n_r^2}{(\nu_r^2-\nu_s^2)^2}\, \frac{C_r C_{r'}}{2}\, \frac{a_s a_{s'}}{\gamma_{sr}\gamma_{s'r'}},$$

$$\dot\phi_q\dot\phi_{s'} = -\tfrac{1}{2}\, C_q C_{q'} \frac{\nu_{q'}^2 n_{q'}^2}{\nu_{q'}^2-\nu_{s'}^2}\, \frac{a_{s'}}{\gamma_{s'q'}} - \tfrac{1}{2}\, C_{s'} C_s \frac{\nu_s^2 n_s^2}{\nu_s^2-\nu_q^2}\, \frac{a_q}{\gamma_{qs}}$$
$$+ \Sigma \frac{C_r C_{r'}}{2}\, \frac{\nu_r^2 \nu_{r'}^2 n_r^2}{(\nu_r^2-\nu_q^2)(\nu_{r'}^2-\nu_{s'}^2)}\, \frac{a_q a_{s'}}{\gamma_{qr}\gamma_{s'r'}}.$$

Here we have supposed $n_s = n_{s'}$. Now we have by equation (14),

$$\frac{1}{n_s^2} = \frac{1}{\nu_s^2} - \Sigma \frac{a_s a_r}{\gamma_{sr}^2}\, \frac{1}{\nu_s^2-\nu_r^2}.$$

Hence in order that $n_{s'} = n_s$, it is as a rule not sufficient that $a_s = a_{s'}$ and $\nu_s = \nu_{s'}$, but we must also have $\gamma_{sr} = \gamma_{s'r'}$, or the structure of the two molecules, at least as far as inter-atomic mean distances are concerned, must be alike. Assuming this to be true, we can rewrite the above values so far as terms of the order $(a/\gamma)^2$, as

$$\dot\phi_s\dot\phi_{s'} = \tfrac{1}{2}\, C_s C_{s'} \nu_s^2 - \tfrac{1}{2}\, C_s C_{s'} \overset{(r \text{ all values but } s.)}{\Sigma} \frac{\nu_s^4}{\nu_r^2-\nu_s^2}\, \frac{a_s a_r}{\gamma_{sr}^2} + \tfrac{1}{2}\Sigma C_r C_{r'} \frac{\nu_r^6}{(\nu_r^2-\nu_s^2)^2}\left(\frac{a_s}{\gamma_{sr}}\right)^2,$$

$$\dot\phi_q\dot\phi_{s'} = -\tfrac{1}{2}\, C_q C_{q'} \frac{\nu_q^4}{\nu_q^2-\nu_s^2}\, \frac{a_s}{\gamma_{sq}} - \tfrac{1}{2}\, C_{s'} C_s \frac{\nu_s^4}{\nu_s^2-\nu_q^2}\, \frac{a_q}{\gamma_{sq}}$$
$$+ \tfrac{1}{2}\Sigma C_r C_{r'} \frac{\nu_r^6}{(\nu_r^2-\nu_q^2)(\nu_r^2-\nu_s^2)}\, \frac{a_q a_s'}{\gamma_{qr}\gamma_{sr}}.$$

Now the corresponding terms in the force function of the two molecules will be

$$U = \frac{1}{4\pi\rho}\Sigma \frac{\dot\phi_s\dot\phi_{s'}}{\gamma_{ss'}} + \frac{1}{4\pi\rho}\Sigma \frac{\dot\phi_q\dot\phi_{s'}}{\gamma_{qs'}}.$$

Let χ equal the angle between the directions of $\gamma_{ss'}$ and γ_{sq}, then

$$\gamma_{qs'}^2 = \gamma_{ss'}^2 + \gamma_{qs}^2 - 2\gamma_{ss'}\gamma_{qs}\cos\chi,$$
$$\frac{1}{\gamma_{qs'}} = \frac{1}{\gamma_{ss'}}\left\{1 + \left(\frac{\gamma_{qs}}{\gamma_{ss'}}\right)P_1(\cos\chi) + \left(\frac{\gamma_{qs}}{\gamma_{ss'}}\right)^2 P_2(\cos\chi) + \dots\right\},$$

where $P_r(\cos\chi) =$ the r^{th} Lagrangean coefficient.

Similarly,

$$\frac{1}{\gamma_{qq'}} = \frac{1}{\gamma_{q's}}\left\{1 + \left(\frac{\gamma_{s'q'}}{\gamma_{q's}}\right)P_1(\cos\chi') + \left(\frac{\gamma_{s'q'}}{\gamma_{q's}}\right)^2 P_2(\cos\chi') + \dots\right\}$$
$$= \frac{1}{\gamma_{ss'}}\left\{1 + \left(\frac{\gamma_{sq}}{\gamma_{ss'}}\right)(P_1(\cos\chi) + P_1(\cos\chi')) + \text{higher terms}\right\},$$

where $\chi' =$ the angle between $\gamma_{s'q'}$ and $\gamma_{s'q} =$ angle between $\gamma_{s'q'}$ and $\gamma_{s's}$ to a first approximation. Or, if we wish to retain terms of the order $[1/m]^2$, we might write

$$\frac{1}{\gamma_{qq'}} = \frac{1}{\gamma_{gg'}}\left\{1 + \frac{\gamma_{gq}}{\gamma_{gg'}}(\cos\chi_q + \cos\chi_{q'})\right\}$$
$$\frac{1}{\gamma_{qs'}} = \frac{1}{\gamma_{gg'}}\left\{1 + \frac{\gamma_{gq}}{\gamma_{gg'}}\cos\chi_q + \frac{\gamma_{g's'}}{\gamma_{gg'}}\cos\chi_{s'}\right\},$$

where g and g' are any two corresponding points in the two molecules, which may or may not be the positions of two like atoms s and s' (e. g. the centroids of the molecules) and χ_q is the angle between $\gamma_{gg'}$ and γ_{gq}, $\chi_{s'}$ between $\gamma_{g'g}$ and $\gamma_{g's'}$. *Thus we see that when we do not neglect terms of the order* $[1/m]^2$, *the force between two molecules is a function of aspect, as well as central distance.*

If this be true for bodies in a very diffused state, it will be still more appreciable when we deal with solid bodies. It follows that the law of inter-molecular force adopted by Poisson, Navier and Cauchy, namely, that inter-molecular force is a function only of central distance, ceases to be true for ether squirt molecules. This might suggest that the rari-constant theory of elasticity can never be true, owing to the action of 'aspect,' even if we neglect the still more important influence of 'modifying action.' But it must be remembered that the stresses with which we have to deal are the summations of a great number of isolated inter-molecular forces, and thus *unless there is any symmetrical distribution of molecules with regard to aspect*, the mean values of $\cos\chi_q$ and $\cos\chi_{s'}$, as well as the mean values of $\cos\chi_q$ and $\cos\chi_{s'}$, will disappear. The terms of the order $[1/m]^3$ would thus contain no aspect influence; indeed we should have to

go to terms of the order $[1/m]^4$ to find them. Thus, at least in a rarefied gas the aspect influence would very probably be negligible. Like remarks apply in a lesser extent to the case of a solid, so that, as far as aspect influence is concerned, we might more reasonably treat a body of 'confused crystallisation'—e. g. many of the metals of construction—as a rari-constant solid, and accordingly give it only one elastic constant, than give a crystal 15 instead of 21 elastic constants.*

After this digression, I return to the above expression for the force function and write down its value as far as $[1/m]^2$. We find

$$V = \frac{\mu_1}{\gamma_{gg'}} + \frac{\mu_2}{\gamma_{gg'}^2}, \tag{41}$$

where

$$\mu_1 = \frac{1}{8\pi\rho}\sum_s C_s C_{s'} \nu_s^2 \Big[1 - \sum_r \frac{\nu_s^2}{\nu_s^2 - \nu_r^2}\,\frac{a_r}{\gamma_{sr}}\Big(2 - \frac{a_s}{\gamma_{sr}}\Big) + \sum_{r,q}\frac{\nu_s^4}{(\nu_s^2-\nu_r^2)(\nu_s^2-\nu_q^2)}\,\frac{a_r a_q}{\gamma_{sr}\gamma_{sq}}\Big],$$

$\sum_s$ denoting a summation with regard to all values of s for the first molecule, s' being the corresponding value for the second; $\sum_r$, a summation with regard to all values of r except $r = s$; and $\sum_{r,q}$, a summation with regard to *all* values of r and q except s, each term except $r = q$ occurring twice or the whole may be written

$$\Big\{\sum \frac{\nu_s^2}{\nu_s^2 - \nu_r^2}\,\frac{a_r}{\gamma_{sr}}\Big\}^2.$$

Thus we may put:

$$\mu_1 = \frac{1}{8\pi\rho}\sum_s C_s C_{s'} \nu_s^2 \Big[\Big(1 - \sum_r \frac{\nu_s^2}{\nu_s^2 - \nu_r^2}\,\frac{a_r}{\gamma_{sr}}\Big)^2 + \sum_r \frac{\nu_s^2}{\nu_s^2 - \nu_r^2}\,\frac{a_r a_s}{\gamma_{sr}^2}\Big]. \tag{42}$$

Further, we have for μ_2:

$$\mu_2 = \frac{1}{8\pi\rho}\sum_s C_s C_{s'}\Big[\gamma_{gs}(\cos\chi_s + \cos\chi_{s'}) - \sum_r \frac{\nu_s^4}{\nu_s^2 - \nu_r^2}\,\frac{a_r}{\gamma_{sr}}(\gamma_{gr}\cos\chi_r + \gamma_{g's'}\cos\chi_{s'})$$
$$- \sum_{r'} \frac{\nu_s^4}{\nu_s^2 - \nu_{r'}^2}\,\frac{a_{r'}}{\gamma_{r's'}}(\gamma_{gs}\cos\chi_s + \gamma_{g'r'}\cos\chi_{r'})\Big],$$

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*I have treated the problems of 'modified action' and 'molecular aspect' at some length in the *History of Elasticity*. See Vol. I, Arts. 921–931, particularly the last article, Art. 1527 (Jellett's views), and Vol. II, Art. 276 (where I have endeavored to show that the Boscovichian atom does not exclude the possibility of multi-constancy, a result confirmed by the 'ether squirt,' which is Boscovichian), and Arts. 304–6 (where I have criticised Saint-Venant's views in the light of 'aspect' and 'modified action'.)

or,
$$\left.\begin{aligned}\mu_2 = \frac{1}{8\pi\rho}\sum_s C_s C_{s'}\Big[\gamma_{gs}(\cos\chi_s + \cos\chi_{s'})\Big(1 - \sum_r \frac{\nu_s^4}{\nu_s^2 - \nu_r^2}\,\frac{a_r}{\gamma_{rs}}\Big) \\ - \sum_{r,r'} \frac{\nu_s^4}{\nu_s^2 - \nu_r^2}\,\frac{a_r}{\gamma_{sr}}\gamma_{gr}(\cos\chi_r + \cos\chi_{r'})\Big]\end{aligned}\right\}, \tag{43}$$

where $\sum_s$ denotes a summation with regard to all values of s, $\sum_r$ with regard to all values of r except $r = s$, and $\sum_{r,r'}$ with regard to all values of r (r' being given the corresponding value for the second molecule)—except those corresponding to $r = s$, $r' = s'$.

Equations (41)–(43) lead us to the following results:

The force between two molecules of a substance, which are at such distances that $[1/m]$ may be neglected as compared with $[1/a]^3$ is, on the hypothesis of equal phases—

a). Partly attractive, varying as the inverse square. This follows from the fact that the first term in the square brackets of μ_1 is positive, and this first term is by far the greater.

b). Partly varying as the inverse cube. This latter term in the force depends upon the influence of aspect, and its mean value would probably be zero for an amorphic body. In such a body it might influence the action between two molecules for a time, but it would not be sensible when we deal with the average of large numbers of molecules.

c). There is no modifying action between any third molecule and any other pair.

As we have supposed the molecules at a very great distance as compared with the atomic distances, it may be doubted whether these results would apply to any substance but a rarefied gas, and the first part of the inter-molecular force will then be very slight, and the second probably quite insensible.

If the phases only differ very slightly we shall obtain the same results by replacing $C_s C_{s'}$ by $C_s C_{s'} \cos(\alpha_s - \alpha_{s'})$, $\alpha_s - \alpha_{s'}$ being now a small angle and its cosine positive.

Sub-case (b). *Unequal Phase.*

The action between any pair of molecules will be the same as before, but $C_s C_{s'}$ must be replaced by $C_s C_{s'} \cos(\alpha_s - \alpha_{s'})$, and this latter expression may be either positive or negative. Thus there would be a tendency between some

molecules to cohere but in others to mutual repulsion. Cohesion of the material as a whole would be impossible. Thus this tendency of some molecules to repel each other might be of service in the explanation of diffusive and evaporative phenomena. I do not know that there is any physical reason against its possibility in the cases of gas or liquid.

18). Case (ii). Suppose $[1/m]$ of the same order as $[1/a]^3$. Now the periods n_r, $n_{r'}$, $n_{r''}$ will all be modified *equally* by the influence of kin-atoms. This variation is given by equation (17) for $\delta\mu$. We have

$$\left.\begin{aligned} 1/n_r^2 &= 1/\nu_r^2 - \mu_r - \delta\mu_r \\ &= 1/\nu_r^2 - \sum_s \frac{a_r a_s}{\gamma_{rs}^2} \frac{1}{\nu_r^2 - \nu_s^2} - \left\{ \frac{\mu_r}{\nu_r^4} \left(\frac{a_{r'} a_{r''}}{\gamma_{r'r''}^2} + \frac{a_r a_{r''}}{\gamma_{rr''}^2} + \frac{a_r a_{r'}}{\gamma_{rr'}^2} \right) \right\}^{\frac{1}{2}}, \end{aligned}\right\} \tag{44}$$

where Σ denotes a summation with regard to all values of s but r.

This expression for $1/n_r^2$ is the same for each of the three molecules.

Returning to equations (39)–(40), we have to calculate the mean values of such types as $\dot{\phi}_s\dot{\phi}_{r'}$ and $\dot{\phi}_q\dot{\phi}_{s'}$. With some reductions I find

$$\left.\begin{aligned} \dot{\phi}_s\dot{\phi}_{s'} = {} & \frac{n_s^2 C_s C_{s'}}{2}\cos(\alpha_s - \alpha_{s'}) + \frac{n_s^2 \varkappa_s' C_s^2}{2} + \frac{n_s^2 \varkappa_{s'} C_{s'}^2}{2} + \frac{n_s^2 \varkappa_{s''}' C_s C_{s''}}{2}\cos(\alpha_s - \alpha_{s''}) \\ & + \frac{n_s^2 \varkappa_{s''} C_{s'} C_{s''}}{2}\cos(\alpha_{s'} - \alpha_{s''}) + \frac{n_s^2 \varkappa_{s''}\varkappa_{s''}' C_{s''}^2}{2} + \frac{n_s^2 \varkappa_{s'}\varkappa_s' C_s C_{s'}}{2}\cos(\alpha_s - \alpha_{s'}) \\ & + \frac{n_s^2 \varkappa_{s'}\varkappa_{s''}' C_{s'} C_{s''}}{2}\cos(\alpha_{s'} - \alpha_{s''}) + \frac{n_s^2 \varkappa_s'\varkappa_{s''} C_s C_{s''}}{2}\cos(\alpha_s - \alpha_{s''}) \\ & + \sum_{r,r'} \frac{n_r^2 \nu_r^4}{(\nu_r^2 - \nu_s^2)^2} \frac{a_s a_{s'}}{\gamma_{sr}\gamma_{s'r'}} \frac{C_r C_{r'}}{2}\cos(\alpha_r - \alpha_{r'}) \\ & - \sum_r \frac{n_r^2 \nu_r^2}{\nu_r^2 - \nu_s^2} \frac{a_s}{\gamma_{sr}} \lambda_{s'r}' \frac{C_r^2}{2} - \sum_{r'} \frac{n_r^2 \nu_{r'}^2}{\nu_{r'}^2 - \nu_{s'}^2} \frac{a_{s'}}{\gamma_{s'r'}} \lambda_{sr'} \frac{C_{r'}^2}{2} \\ & + \sum_{r,r'} n_r^2 \lambda_{sr'}\lambda_{s'r}' \frac{C_r C_{r'}}{2}\cos(\alpha_r - \alpha_{r'}) + \sum_{r,r''} n_r^2 \lambda_{sr''}\lambda_{s'r''}' \frac{C_r''}{2} \\ & - \sum_{r,r''} \frac{n_r^2 \nu_r^2}{\nu_r^2 - \nu_s^2} \frac{a_s}{\gamma_{sr}} \lambda_{s'r''}' C_r C_{r''}\cos(\alpha_r - \alpha_{r''}) \\ & - \sum_{r',r''} \frac{n_r^2 \nu_{r'}^2}{\nu_{r'}^2 - \nu_{s'}^2} \frac{a_{s'}}{\gamma_{s'r'}} \lambda_{sr''} \frac{C_{r'} C_{r''}}{2}\cos(\alpha_{r'} - \alpha_{r''}) \\ & + \sum_{r',r''} n_r^2 \lambda_{sr'}\lambda_{s'r''} \frac{C_{r'} C_{r''}}{2}\cos(\alpha_{r'} - \alpha_{r''}) + \sum_{r,r''} n_r^2 \lambda_{s'r}'\lambda_{sr''} \frac{C_r C_{r''}}{2}\cos(\alpha_r - \alpha_{r''}^2) \end{aligned}\right\} \tag{45}$$

Here $\sum_r$ is a summation with regard to all the atoms of the first molecule except those which make the subject of the summation infinite. $\sum_{r,r'}$ signifies a

summation for kin-atoms of the first and second molecule with the like restriction. The other symbols have corresponding meanings.

$$\left.\begin{aligned}
\dot\phi_q\dot\phi_{s'} = &-\frac{n_s^2\nu_s^2}{\nu_s^2-\nu_q^2}\,\frac{a_q}{\gamma_{qs}}\,\frac{C_{s'}C_s}{2}\cos(\alpha_{s'}-\alpha_s)\\
&-\frac{n_q^2\nu_{q'}^2}{\nu_{q'}^2-\nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{s'q'}}\,\frac{C_{q'}C_q}{2}\cos(\alpha_q-\alpha_{q'})\\
&+\sum_{r,r'}\frac{n_r^2\nu_r^4}{(\nu_r^2-\nu_{s'}^2)(\nu_r^2-\nu_q^2)}\,\frac{a_{s'}}{\gamma_{s'r'}}\,\frac{a_q}{\gamma_{qr}}\,\frac{C_rC_{r'}}{2}\cos(\alpha_r-\alpha_{r'})\\
&-\frac{\nu_s^2n_s^2}{\nu_s^2-\nu_q^2}\,\frac{a_q}{\gamma_{qs}}\,\varkappa_s'\,\frac{C_s^2}{2}-\frac{\nu_{q'}^2n_q^2}{\nu_{q'}^2-\nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{q's'}}\,\varkappa_{q'}\,\frac{C_{q'}^2}{2}\\
&-\frac{n_q^2\nu_{q'}^2}{\nu_{q'}^2-\nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{q's'}}\,\varkappa_{q''}\,\frac{C_{q'}C_{q''}}{2}\cos(\alpha_{q'}-\alpha_{q''})\\
&-\frac{n_s^2\nu_s^2}{\nu_s^2-\nu_q^2}\,\frac{a_q}{\gamma_{qs}}\,\varkappa_{s''}'\,\frac{C_{s''}C_s}{2}\cos(\alpha_s-\alpha_{s''})\\
&+n_s^2\lambda_{qs'}\frac{C_{s'}^2}{2}+n_q^2\lambda_{s'q}'\frac{C_q^2}{2}+n_s^2\lambda_{qs''}\frac{C_{s''}^2}{2}\varkappa_{s''}'+n_q^2\lambda_{s'q''}'\frac{C_{q''}^2}{2}\varkappa_{q''}\\
&+n_s^2\lambda_{qs''}\frac{C_{s'}C_{s''}}{2}\cos(\alpha_{s'}-\alpha_{s''})+n_q^2\lambda_{s'q''}'\frac{C_qC_{q''}}{2}\cos(\alpha_q-\alpha_{q''})\\
&+n_s^2\varkappa_s'\lambda_{qs'}\frac{C_sC_{s'}}{2}\cos(\alpha_s-\alpha_{s'})+n_q^2\varkappa_{q'}\lambda_{s'q}'\frac{C_qC_{q'}}{2}\cos(\alpha_q-\alpha_{q'})\\
&+n_s^2\varkappa_s'\lambda_{qs''}\frac{C_sC_{s''}}{2}\cos(\alpha_s-\alpha_{s''})+n_q^2\varkappa_{q''}\lambda_{s'q}'\frac{C_qC_{q''}}{2}\cos(\alpha_q-\alpha_{q''})\\
&+n_s^2\varkappa_{s''}'\lambda_{qs'}\frac{C_{s''}C_{s'}}{2}\cos(\alpha_{s''}-\alpha_{s'})+n_q^2\varkappa_{q'}\lambda_{s'q''}'\frac{C_{q'}C_{q''}}{2}\cos(\alpha_{q'}-\alpha_{q''})\\
&-\sum_{r'}\frac{n_{r'}^2\nu_{r'}^2}{\nu_{r'}^2-\nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{s'r'}}\,\lambda_{qr'}\,\frac{C_{r'}^2}{2}-\sum_r\frac{n_r^2\nu_r^2}{\nu_r^2-\nu_q^2}\,\frac{a_q}{\gamma_{qr}}\,\lambda_{s'r}'\,\frac{C_r^2}{2}\\
&-\sum_{r',r''}\frac{n_r^2\nu_r^2}{\nu_{r'}^2-\nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{s'r'}}\,\lambda_{qr''}\,\frac{C_{r'}C_{r''}}{2}\cos(\alpha_{r'}-\alpha_{r''})\\
&-\sum_{r,r''}\frac{n_r^2\nu_r^2}{\nu_r^2-\nu_q^2}\,\frac{a_q}{\gamma_{qr}}\,\lambda_{s'r''}'\,\frac{C_rC_{r''}}{2}\cos(\alpha_r-\alpha_{r''})\\
&+\sum_{r,r'}\lambda_{s'r}'\lambda_{qr'}\frac{C_rC_{r'}}{2}\cos(\alpha_r-\alpha_{r'})+\sum_{r,r''}\lambda_{qr''}\lambda_{s'r}'\frac{C_rC_{r''}}{2}\cos(\alpha_r-\alpha_{r''})\\
&+\sum_{r',r''}\lambda_{s'r''}'\lambda_{qr'}\frac{C_{r'}C_{r''}}{2}\cos(\alpha_{r'}-\alpha_{r''})+\sum_{r''}\lambda_{s'r''}'\lambda_{qr''}\frac{C_{r''}^2}{2}
\end{aligned}\right\}. \qquad (46)$$

The summations are of the same nature as in (45). These formulae appear to give extremely complex results for the types of force function terms due to kin- and non-kin-atoms in different molecules. But this complexity is very

much simplified if we only retain the more important terms.* In order to do this we must determine the order of the κ's and λ's on the supposition that $[1/m] = [1/a]^3$; in this case the types for κ and λ will be those of equations (35) and (36), and $\delta\mu$ is of the order $[1/a]^{\frac{8}{3}}$.

$$\begin{array}{ll} \kappa_{s'}\,,\ \kappa_{s''}\,,\ \kappa'_{s}\,,\ \kappa'_{s''} & \text{are of the order } [1/a]^{\frac{1}{3}}, \\ \lambda_{sr'},\ \lambda_{sr''},\ \lambda'_{s'r},\ \lambda'_{s'r''} & \text{"} \quad \text{"} \qquad \text{"} \qquad [1/a]^{\frac{4}{3}}. \end{array}$$

Thus, if we only retain terms in the expressions $\dot{\phi}_s\dot{\phi}_{s'}$ and $\dot{\phi}_q\dot{\phi}_{s'}$ so far as the order $[1/a]^2$, this will mean retaining terms up to $[1/a]^5$ in the force function and up to $[1/a]^6$ in the force. We see at once that the terms involving the λ's are all either of the order $[1/a]^{\frac{7}{3}}$ or $[1/a]^{\frac{8}{3}}$ and may be neglected in $\dot{\phi}_s\dot{\phi}_{s'}$. In $\dot{\phi}_q\dot{\phi}_{s'}$ we shall still, however, have to keep the long series of terms with products of κ's and λ's as these are of the order $[1/a]^{\frac{5}{3}}$. Further, since $\delta\mu$ is of the order $[1/a]^{\frac{8}{3}}$, we need only substitute for n_s^2, $\nu_s^2(1 + \nu_s^2\mu_s)$, and this in the first term of $\dot{\phi}_s\dot{\phi}_{s'}$ only.

We find

$$\left.\begin{aligned} \dot{\phi}_s\dot{\phi}_{s'} = {} & \frac{\nu_s^2 C_s C_{s'}}{2}\cos(\alpha_s - \alpha_{s'})\left\{1 + \sum_r \frac{a_s a_r}{\gamma_{sr}^2}\,\frac{\nu_s^2}{\nu_s^2 - \nu_r^2}\right\} \\ & - \frac{a_{s'}}{\gamma_{ss'}}\,\frac{1}{\delta\mu_s}\,\frac{C_s^2}{2} - \frac{a_s}{\gamma_{ss'}}\,\frac{1}{\delta\mu_{s'}}\,\frac{C_{s'}^2}{2} \\ & - \frac{a_{s'}}{\gamma_{s''s'}}\,\frac{1}{\delta\mu_{s''}}\,\frac{C_s C_{s''}}{2}\cos(\alpha_s - \alpha_{s''}) - \frac{a_s}{\gamma_{ss''}}\,\frac{1}{\delta\mu_{s''}}\,\frac{C_{s'} C_{s''}}{2}\cos(\alpha_{s'} - \alpha_{s''}) \\ & + \frac{a_s}{\gamma_{ss''}}\,\frac{a_{s'}}{\gamma_{s's''}}\,\frac{1}{\nu_s^2}\,\frac{1}{(\delta\mu_{s''})^2}\,\frac{C_{s''}^2}{2} + \frac{a_s a_{s'}}{\gamma_{ss'}^2}\,\frac{1}{\nu_s^2}\,\frac{1}{\delta\mu_s\delta\mu_{s'}}\,\frac{C_{s'} C_{s''}}{2}\cos(\alpha_{s'} - \alpha_{s''}) \\ & + \frac{a_s}{\gamma_{ss'}}\,\frac{a_{s'}}{\gamma_{s's''}}\,\frac{1}{\nu_s^2}\,\frac{1}{\delta\mu_{s'}\delta\mu_{s''}}\,\frac{C_{s'} C_{s''}}{2}\cos(\alpha_{s'} - \alpha_{s''}) \\ & + \frac{a_s}{\gamma_{ss''}}\,\frac{a_{s'}}{\gamma_{ss'}}\,\frac{1}{\nu_s^2}\,\frac{1}{\delta\mu_{s''}\delta\mu_s}\,\frac{C_s C_{s''}}{2}\cos(\alpha_s - \alpha_{s''}) \\ & + \sum_{r,r'} \frac{\nu^6}{(\nu_r^2 - \nu_s^2)^2}\,\frac{a_s a_{s'}}{\gamma_{sr}\gamma_{s'r'}}\,\frac{C_r C_{r'}}{2}\cos(\alpha_r - \alpha_{r'}) \end{aligned}\right\}. \quad (47)$$

This result shows us that if we were only to retain terms of the order $[1/a]^{\frac{4}{3}}$, the

* The terms neglected may correspond to important physical or chemical phenomena, but at present we must seek only the more general results of the theory. The influence of one molecule on another will in itself be as complex as the action of one planetary system on another supposed in its neighborhood; the complete disturbing action of the whole system on one planet would be not more difficult to determine than that of the action of the two molecules on one of their atoms.

non-kin-atoms would have no influence at all on the force between kin-atoms, for the terms they introduce are only of the order $[1/a]^4$.

Further we find:

$$
\left.
\begin{aligned}
\dot{\phi}_q\dot{\phi}_{s'} = & -\frac{\nu_s^4}{\nu_s^2-\nu_q^2}\,\frac{a_q}{\gamma_{qs}}\,\frac{C_{s'}C_s}{2}\cos(\alpha_{s'}-\alpha_s) \\
& -\frac{\nu_{q'}^4}{\nu_{q'}^2-\nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{s'q'}}\,\frac{C_{q'}C_q}{2}\cos(\alpha_q-\alpha_{q'}) \\
& +\Sigma\frac{\nu_r^6}{(\nu_{r'}^2-\nu_{s'}^2)(\nu_r^2-\nu_q^2)}\,\frac{a_{s'}}{\gamma_{s'r'}}\,\frac{a_q}{\gamma_{qr}}\,\frac{C_rC_{r'}}{2}\cos(\alpha_r-\alpha_{r'}) \\
& +\frac{\nu_s^2}{\nu_s^2-\nu_q^2}\,\frac{a_q}{\gamma_{qs}}\,\frac{a_{s'}}{\gamma_{ss'}}\,\frac{1}{\delta\mu_s}\,\frac{C_s^2}{2}+\frac{\nu_q^2}{\nu_{q'}^2-\nu_{s'}^2}\,\frac{a_s}{\gamma_{q's'}}\,\frac{a_q}{\gamma_{qq'}}\,\frac{1}{\delta\mu_q}\,\frac{C_{q'}^2}{2} \\
& +\frac{\nu_{q'}^2}{\nu_{q'}^2-\nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{q's'}}\,\frac{a_q}{\gamma_{qq''}}\,\frac{1}{\delta\mu_{q''}}\,\frac{C_qC_{q''}}{2}\cos(\alpha_{q'}-\alpha_{q''}) \\
& +\frac{\nu_s^2}{\nu_s^2-\nu_q^2}\,\frac{a_q}{\gamma_{qs}}\,\frac{a_{s'}}{\gamma_{s''s'}}\,\frac{1}{\delta\mu_{s''}}\,\frac{C_{s''}C_s}{2}\cos(\alpha_s-\alpha_{s''}) \\
& +\frac{\nu_s^2}{\nu_s^2-\nu_q^2}\,\frac{a_q}{\gamma_{qs}}\,\frac{a_s}{\gamma_{ss'}}\,\frac{1}{\delta\mu_{s'}}\,\frac{C_{s'}^2}{2}+\frac{\nu_q^2}{\nu_{q'}^2-\nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{q's'}}\,\frac{a_{q'}}{\gamma_{qq'}}\,\frac{1}{\delta\mu_q}\,\frac{C_q^2}{2} \\
& +\frac{\nu_s^2}{\nu_s^2-\nu_q^2}\,\frac{a_q}{\gamma_{sq}}\,\frac{a_s}{\gamma_{ss''}}\,\frac{1}{\delta\mu_{s''}}\,\frac{C_{s'}C_{s''}}{2}\cos(\alpha_{s'}-\alpha_{s''}) \\
& +\frac{\nu_q^2}{\nu_{q'}^2-\nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{q's'}}\,\frac{a_{q'}}{\gamma_{q'q''}}\,\frac{1}{\delta\mu_{q''}}\,\frac{C_qC_{q''}}{2}\cos(\alpha_q-\alpha_{q''}) \\
& -\frac{1}{\nu_s^2-\nu_q^2}\,\frac{a_q}{\gamma_{sq}}\,\frac{a_s}{\gamma_{ss''}}\,\frac{a_{s'}}{\gamma_{s's''}}\,\frac{1}{\delta\mu_{s''}^2}\,\frac{C_{s''}^2}{2} \\
& -\frac{1}{\nu_{q'}^2-\nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{q's'}}\,\frac{a_{q'}}{\gamma_{q'q''}}\,\frac{a_q}{a_{qq''}}\,\frac{1}{\delta\mu_{q''}^2}\,\frac{C_{q''}^2}{2} \\
& -\frac{1}{\nu_{s'}^2-\nu_{q'}^2}\,\frac{a_q}{\gamma_{qs}}\,\frac{a_s}{\gamma_{ss'}}\,\frac{a_{s'}}{\gamma_{ss'}}\,\frac{1}{\delta\mu_s\delta\mu_{s'}}\,\frac{C_sC_{s'}}{2}\cos(\alpha_s-\alpha_{s'}) \\
& -\frac{1}{\nu_{s''}^2-\nu_{q'}^2}\,\frac{a_q}{\gamma_{qs}}\,\frac{a_s}{\gamma_{ss''}}\,\frac{a_{s'}}{\gamma_{ss'}}\,\frac{1}{\delta\mu_s\delta\mu_{s''}}\,\frac{C_sC_{s''}}{2}\cos(\alpha_s-\alpha_{s''}) \\
& -\frac{1}{\nu_{s'}^2-\nu_{q'}^2}\,\frac{a_q}{\gamma_{qs}}\,\frac{a_s}{\gamma_{ss'}}\,\frac{a_{s'}}{\gamma_{s's''}}\,\frac{1}{\delta\mu_{s'}\delta\mu_{s''}}\,\frac{C_{s''}C_{s'}}{2}\cos(\alpha_{s'}-\alpha_{s''}) \\
& -\frac{1}{\nu_{q'}^2-\nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{s'q'}}\,\frac{a_{q'}}{\gamma_{qq'}}\,\frac{a_q}{\gamma_{qq'}}\,\frac{1}{\delta\mu_q\delta\mu_{q'}}\,\frac{C_qC_{q'}}{2}\cos(\alpha_q-\alpha_{q'}) \\
& -\frac{1}{\nu_{q'}^2-\nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{s'q'}}\,\frac{a_{q'}}{\gamma_{qq'}}\,\frac{a_q}{\gamma_{qq''}}\,\frac{1}{\delta\mu_q\delta\mu_{q''}}\,\frac{C_qC_{q''}}{2}\cos(\alpha_q-\alpha_{q''}) \\
& -\frac{1}{\nu_{q'}^2-\nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{s'q'}}\,\frac{a_{q'}}{\gamma_{q'q''}}\,\frac{a_q}{\gamma_{qq'}}\,\frac{1}{\delta\mu_{q'}\delta\mu_{q''}}\,\frac{C_{q'}C_{q''}}{2}\cos(\alpha_{q'}-\alpha_{q''})
\end{aligned}
\right\} \quad (48)
$$

Sub-case (a). *Unequal Phase.*

We suppose that for a great number of molecules the mean difference of phase takes all values from 0 to 2π. Thus although terms like $\cos(\alpha_s - \alpha_{s'})$ may have a finite value for any pair of molecules, yet if we are considering the elastic or cohesive stress across an elementary plane area at a point, such terms will disappear owing to our having to take the mean for a very great number of molecules. We may therefore omit them from the beginning, if we remember that we are not then dealing with any individual pair of molecules, but only with so much of the force between them as will appear after summing for a very great number of molecules. Thus the atoms, or some of them, in one molecule may attract or repel the atoms, or some of them, in a second molecule according as to the magnitude of the difference in phase, but we shall have the following results for $\dot{\phi}_s\dot{\phi}_{s'}$ and $\dot{\phi}_q\dot{\phi}_{s'}$ if we take the mean value for a very great number of molecules of their mean time values as given in (47) and (48)

$$\dot{\phi}_s\dot{\phi}_{s'} = -\frac{a_{s'}}{\gamma_{ss'}}\frac{1}{\delta\mu_s}\frac{C_s^2}{2} - \frac{a_s}{\gamma_{ss'}}\frac{1}{\delta\mu_{s'}}\frac{C_{s'}^2}{2} + \frac{a_s a_{s'}}{\gamma_{ss''}\gamma_{s's''}}\frac{1}{\nu_s^2}\frac{1}{(\delta\mu_{s''})^2}\frac{C_{s''}^2}{2}. \tag{49}$$

$$\left.\begin{aligned}\dot{\phi}_q\dot{\phi}_{s'} = {} & \frac{\nu_s^2}{\nu_s^2-\nu_q^2}\frac{a_q}{\gamma_{qs}}\frac{a_{s'}}{\gamma_{ss'}}\frac{1}{\delta\mu_s}\frac{C_s^2}{2} + \frac{\nu_q^2}{\nu_{q'}^2-\nu_{s'}^2}\frac{a_{s'}}{\gamma_{q's'}}\frac{a_q}{\gamma_{qq'}}\frac{1}{\delta\mu_{q'}}\frac{C_{q'}^2}{2} \\ & + \frac{\nu_s^2}{\nu_s^2-\nu_q^2}\frac{a_q}{\gamma_{qs}}\frac{a_s}{\gamma_{ss'}}\frac{1}{\delta\mu_{s'}}\frac{C_{s'}^2}{2} + \frac{\nu_q^2}{\nu_{q'}^2-\nu_{s'}^2}\frac{a_{s'}}{\gamma_{q's'}}\frac{a_{q'}}{\gamma_{qq'}}\frac{1}{\delta\mu_q}\frac{C_q^2}{2} \\ & - \frac{1}{\nu_s^2-\nu_q^2}\frac{a_q}{\gamma_{qs}}\frac{a_s}{\gamma_{ss''}}\frac{a_{s'}}{\gamma_{s's''}}\frac{1}{(\delta\mu_{s''})^2}\frac{C_{s''}^2}{2} \\ & - \frac{1}{\nu_{q'}^2-\nu_{s'}^2}\frac{a_{s'}}{\gamma_{q's'}}\frac{a_{q'}}{\gamma_{q'q''}}\frac{a_q}{\gamma_{qq''}}\frac{1}{(\delta\mu_{q''})^2}\frac{C_{q''}^2}{2}\end{aligned}\right\}. \tag{50}$$

The terms in the expression for $\dot{\phi}_s\dot{\phi}_{s'}$ are of the order $[1/a]^{\frac{1}{3}}$ and $[1/a]^{\frac{2}{3}}$, while those in that for $\dot{\phi}_q\dot{\phi}_{s'}$ $[1/a]^{\frac{4}{3}}$ and $[1/a]^{\frac{5}{3}}$. The former gives rise to the more important terms in the force function. Thus the most important terms of inter-molecular force are given by the force function

$$\left.\begin{aligned}U = {} & -\sum_{s,s',s''}\frac{1}{8\pi\rho}\frac{a_s}{\gamma_{ss'}^2}(C_s^2+C_{s'}^2)\frac{1}{\left\{\frac{\mu_s}{\nu_s^4}\left(\frac{a_s^2}{\gamma_{s's''}^2}+\frac{a_s^2}{\gamma_{ss''}^2}+\frac{a_s^2}{\gamma_{ss'}^2}\right)\right\}^{\frac{1}{3}}} \\ & + \sum_{s,s',s''}\frac{1}{8\pi\rho}\frac{a_s^2}{\gamma_{ss'}\gamma_{ss''}\gamma_{s's''}}C_{s''}^2\frac{1}{\left\{\frac{\mu_s}{\nu_s^4}\left(\frac{a_s^2}{\gamma_{s's''}^2}+\frac{a_s^2}{\gamma_{ss''}^2}+\frac{a_s^2}{\gamma_{ss'}^2}\right)\right\}^{\frac{2}{3}}}\end{aligned}\right\}, \tag{51}$$

To free this result from the great variety of inter-atomic distances we ought to replace $1/\gamma_{ss'}$, etc., by expressions in terms of $1/\gamma_{gg'}$, etc., as in Art. 17, but we may replace $1/\gamma_{ss'}$ straight away by $1/\gamma_{gg'}$ if we neglect terms of the order $[1/a]^{\frac{12}{3}}$, retaining those of the order $[1/a]^{\frac{14}{3}}$. Thus we can rewrite the above expression:

$$\left.\begin{aligned} U = & -\frac{1}{8\pi\rho}\,\frac{1}{\gamma_{gg'}^2}\,\frac{1}{\left\{\dfrac{1}{\gamma_{g'g''}^2}+\dfrac{1}{\gamma_{gg''}^2}+\dfrac{1}{\gamma_{gg'}^2}\right\}^{\frac{1}{3}}}\sum_{s,s'}(C_s^2+C_{s'}^2)\left(\frac{\nu_s^4 a_s}{\mu_s}\right)^{\frac{1}{3}} \\ & +\frac{1}{8\pi\rho}\,\frac{1}{\gamma_{gg'}\gamma_{gg''}\gamma_{g'g''}}\,\frac{1}{\left\{\dfrac{1}{\gamma_{g'g''}^2}+\dfrac{1}{\gamma_{gg''}^2}+\dfrac{1}{\gamma_{gg'}^2}\right\}^{\frac{2}{3}}}\sum_{s,s''}\left\{\frac{\nu_s^4 a_s}{\mu_s}\right\}^{\frac{2}{3}} C_{s''}^2 \end{aligned}\right\} \tag{52}$$

I will now note the conclusions we can draw from this approximation to the value of U.

a). 'Aspect' does not influence inter-molecular action, at least in the principal terms. Indeed, although we have indicated that it would occur in terms of the order $[1/a]^{\frac{12}{3}}$, yet even here in the case of an amorphic body it would disappear for the mean of a great number of molecules.

b). The modifying effect of the presence of a third molecule on the law of inter-molecular force between two others is not merely a feature of the second or higher approximations, but is fundamental to the first approximation; i. e. so long as the third molecule is at distances from the first two of the same order as their distance, modifying action cannot be neglected.

c). The first term in the force function is of the order $[1/a]^{\frac{10}{3}}$ and *negative*, hence the corresponding force is repulsive; the second term is of the order $[1/a]^{\frac{11}{3}}$, and so it might thus appear that two molecules must always repel each other. But it must be remembered that we have only taken the second term for a *single modifying molecule,* and that for the cohesion the second term would have to be summed over the whole sphere of inter-molecular influence. This summation would cause the term to rise in order, and we thus see that, *if on the hypothesis of unequal phases the force between two molecules be attractive, then this attraction is solely due to the truth of the hypothesis of modified action;* i. e. cohesion depends on the truth of this hypothesis.

d). While we thus see that the chief terms in U depend on the action of kin-atoms, we have also noted that kin-atoms tend to produce a repulsive inter-molecular action, and unless their influence is counteracted by the modifying

action of other molecules, we have a repulsive force between kin-atoms. If we consider the terms (which are of a higher order) in the force between non-kin-atoms and suppose the molecules absolutely alike so that $a_s = a_{s'}$, $C_s = C_{s'}$, etc., we find

$$\frac{\dot{\phi}_q \dot{\phi}_s}{\gamma_{qs'}} = \frac{a_q a_s}{\gamma_{qs}} \frac{1}{\gamma_{gg'}^2} \frac{1}{\left\{\frac{1}{\gamma_{g'g''}^2} + \frac{1}{\gamma_{gg''}^2} + \frac{1}{\gamma_{gg'}^2}\right\}^{\frac{1}{3}}} \frac{C_s^2 \left(\frac{\nu_s^{10}}{\mu_s a_s^2}\right)^{\frac{1}{3}} - C_q^2 \left(\frac{\nu_q^{10}}{\mu_q a_q^2}\right)^{\frac{1}{3}}}{\nu_s^2 - \nu_q^2}$$

to a first approximation.

Hence the force between non-kin atoms of different molecules will be attractive if the atom whose free vibration has the shorter period $(\nu_s > \nu_q)$ be of such amplitude in vibrational flow that

$$C_s^2 \left(\frac{\nu_s^{10}}{\mu_s a_s^2}\right)^{\frac{1}{3}} > C_q^2 \left(\frac{\nu_q^{10}}{\mu_q a_q^2}\right)^{\frac{1}{3}}.$$

Now C_s and C_q will probably depend to a very great extent on the period of any energy which may be in a state of transmission through the ether in the neighborhood of the atom. For example, light or heat energy of a period nearly equal to $2\pi/n_s$ would tend greatly to affect C_s but not C_q, and energy of a period nearly equal to $2\pi/n_q$ would affect C_q and not C_s, thus we should expect the inter-molecular action of non-kin atoms to be even capable of changing its sign owing to the nature of the optic, thermal or electric field in which the molecules are placed. Thus these non-kin atom terms in the force function would produce a secondary influence on the general law of cohesion due to the conditions of the optic, thermal or electric field. Of course these conditions would also affect C_s, $C_{s'}$ and $C_{s''}$, but examining the value of U in (51), we see that to decrease or increase these quantities would, speaking very generally, be rather quantitative; i. e. there would not be a distinctly opposite effect on U for a field in which energy of period $2\pi/n_s$ and a field in which energy of period $2\pi/n_q$ were respectively prominent. The *qualitative* effect therefore of energy of different periods would, if it is in any case sensible, have to be attributed to the action of non-kin atoms and the terms in U arising from (50).

e). Since U consists of terms of opposite sign, and we have indicated that owing to the summation which we must make for a great number of modifying molecules, the second term may rise to equal order and importance with the first (see (c)), it follows that for a certain value of $\gamma_{gg'}$ $dU/d\gamma_{gg'}$ is zero, or *the inter-molecular force changes sign.* This gives, then, a mean inter-molecular distance at which for definite values of the C's, i. e. for definite conditions of the thermal, optic and electric fields, the molecules exert no force on each other. If we bring

the molecules closer together they will repel; if we separate them, they will attract. This is the ordinary phenomena of elastic stress, and this change in the sign of *inter-molecular* force seems an important addition to the discussion of cohesion as given in my first paper. It arises from the terms due to kin-atoms and the modifying action, which were neglected in that paper.

f). The presence of a greater or less number of kin-atoms will obviously greatly influence the law of cohesion, although whether they will tend to increase or decrease the mean inter-molecular distance must depend largely on the nature of the second term of U in (51) when we sum it for *all* the modifying molecules. But the mean inter-molecular distance will settle the volume. Thus we should expect remarkable divergences in the volumetric analysis of quantities like CH_4O, C_2H_6O, C_3H_8O or Cu_2O, Cu_2O_2, etc.; and again, noteworthy variations in the elasticity and cohesion of steel as we introduce more and more carbon.

It seems not improbable that some physical measure of the effect of kin-atoms on cohesion might be obtained from the cohesion of plates of different substances when pressed together, the molecules of the two substances having no kin-atoms, or one or more kin-atoms in their structure.

Sub-case (b). *Unequal Phase.*

20). I now pass to the consideration of the value of the terms in the force function when the phases of kin-atoms in different molecules are either equal or differ by less than $\pi/2$. The physical probability seems that if they do not take all possible values they will only differ slightly (see §47 of my first paper), so that we have only to replace the cosines in (47) and (48) by a quantity ε very nearly equal to unity in order to obtain in this sub-case the mean values of $\dot{\phi}_s\dot{\phi}_{s'}$ and $\dot{\phi}_q\dot{\phi}_{s'}$.

If we retain terms up to $[1/a]^2$ we can omit none of those given in (47) for $\dot{\phi}_s\dot{\phi}_{s'}$; if we retain only terms of order $[1/a]^{5/3}$ we may omit the second term in the bracket of first line and the last term; if we retain only terms of the order $[1/a]^{4/3}$ we preserve only the terms in the first to the third lines, omitting second term of the first. Similarly for $\dot{\phi}_q\dot{\phi}_{s'}$ for

approximation up to $[1/a]^2$ we retain all terms.
" " $[1/a]^{5/3}$ " " but those in the third line.
" " $[1/a]^{4/3}$ " 1st, 2d, 4th, 9th lines only.
" " $[1/a]^{2/3}$ " 1st and 2d lines only.
" " $[1/a]^{1/3}$ we need consider no terms in $\dot{\phi}_q\dot{\phi}_{s'}$.

As we only wish to obtain a general idea of inter-molecular force, let us adopt the approximation to the order $[1/a]^{\frac{1}{3}}$. So far as the terms then dealt with are concerned, *aspect* will not enter into our value of the force function U, as it would involve terms of order $[1/a]^5$, whereas we are only retaining those of order $[1/a]^{\frac{10}{3}}$. We find

$$\left.\begin{aligned} U= &\ \frac{1}{8\pi\rho}\,\frac{1}{\gamma_{gg'}}\sum_{s,s'} C_s C_{s'}\,\varepsilon_{ss'} \\ &-\frac{1}{8\pi\rho}\,\frac{1}{\gamma_{gg'}^2}\sum_{s,s's''}\left\{\frac{a_{s'}C_s^2}{\delta\mu_s}+\frac{a_s C_{s'}^2}{\delta\mu_s}\right\} \\ &-\frac{1}{8\pi\rho}\,\frac{1}{\gamma_{gg'}}\sum_{s,s's''}\left\{\frac{a_{s'}C_s C_{s''}\varepsilon_{ss''}}{\gamma_{s''s'}\,\delta\mu_{s''}}+\frac{a_s C_{s'} C_{s''}\varepsilon_{s's''}}{\gamma_{ss''}\,\delta\mu_{s''}}\right\} \end{aligned}\right\} \qquad (53)$$

We may now deduce some general results from this expression:

a). Neglecting modifying action, the inter-molecular force varies as the inverse square* and is attractive. Thus for this case the force of cohesion could never become negative. Turning to the second and third terms, however, these give *repulsive forces.* The order of the part of the force due to the first term will be $[1/a]^6$, while that due to the second and third of the order $[1/a]^{\frac{13}{3}}$. Thus the difference in order is not very great, and if the modifying action as expressed in the third term be summed for a great number of molecules, this term may rise in order and possibly become so great that the force changes sign and becomes repulsive. The possibility of repulsive inter-molecular force would thus again depend on the law of modified action.

b). A distinction must be drawn between (a) of this article and (c) of the preceding article. For unequal phases the term of least order in the force is repulsive and of order $[1/a]^{\frac{13}{3}}$. To account for a cohesive force and the necessary change in sign, we must suppose the attractive terms in the force of order $[1/a]^{\frac{20}{3}}$ to become all-important on summation for a great number of molecules. For equal phases the term of least order in the force is attractive and of order $[1/a]^6$, and the necessary change of sign to explain cohesion is obtained by supposing terms of the order $[1/a]^{\frac{13}{3}}$ to rise into importance on summation. The latter seems a more reasonable hypothesis, as it would allow of the union of two solitary molecules, which the former would not. The former indeed indicates that

* It must not be supposed that because this force varies as the 'inverse square', that cohesion is thus only a part of gravitating force, and so our theory be liable to the destructive criticism of Belli and others. That criticism is based, not on the power of the distance, but on the magnitude of the constant of variation being the same as in the case of gravitation.

only a certain number of molecules would be able to cohere at all unless they were chosen with particular differences of phase. So far as I know, there are no physical facts which would point to any such selective action on the part of molecules when uniting in small groups to form a solid. At the same time the phenomena of smell tell us that bodies must be continually throwing off and therefore *repelling* their external molecules. This must either mean a sufficient difference in phase to produce a repulsive force between molecules, or else that the modifying action on *superficial* molecules is not always sufficiently great to overcome the repulsive influence produced by kin-atoms. Thus we should have to look upon smell on the hypothesis of nearly equal phase as a process by which individual eccentric or 'unsympathetic' molecules—i. e. those whose atoms had a wide divergence of phase from that of the mean of their kin-atoms in the system—were gradually brought to the surface and ejected; but on the hypothesis of unequal phase, smell would be due to the failure of sufficient "modifying action" at the surface.

c). Remarks (a) and (b) of the previous article apply in this case also, while we may treat the law of force between non-kindred atoms by examining the first two terms of (48). For exactly like molecules we have, to a first approximation, the term in the force function for the q^{th} and s'^{th} atoms,

$$-\frac{1}{\gamma_{gg'}}\,\frac{a_q a_s}{\gamma_{sq}}\,\frac{1}{8\pi\rho}\,\frac{C_s^2\nu_s^4/a_s - C_q^2\nu_q^4/a_q}{\nu_s^2-\nu_q^2},$$

or there is an attractive force between the q^{th} and s'^{th} atoms,

$$=\frac{1}{\gamma_{gg'}^2}\,\frac{\pi}{2\rho}\,\frac{1}{\nu_s^2\nu_q^2}\,\frac{a_q a_s}{\gamma_{sq}}\,\frac{\dfrac{C_s^2\nu_s^4}{a_s}-\dfrac{C_q^2\nu_q^4}{a_q}}{\dfrac{4\pi^2}{\nu_s^2}-\dfrac{4\pi^2}{\nu_q^2}}. \tag{54}$$

But this force—

$$=\tfrac{1}{2}\left(\frac{\gamma_{sq}}{\gamma_{gg'}}\right)^2\left\{\begin{array}{l}\text{atomic force between } q^{\text{th}} \text{ and } s^{\text{th}}\\ \text{atoms of one molecule.}\end{array}\right\} \tag{55}$$

This follows by (17), if we only retain the lowest powers of $F_{ss'}$. Thus the intermolecular force between the q^{th} and s'^{th} atoms is at once expressed in terms of the atomic force between q^{th} and s^{th} atoms. It will obviously be attractive or repulsive according to the sign of the "chemical affinity" or the last factor on right of (54). *Thus, ceteris paribus, the more difficult its molecule is to disassociate, the greater the cohesion of the substance.*

21). Let us now pass to the third case when inter-molecular distance bears to inter-atomic the ratio of order given by $[1/m] = [1/a]^2$.

In this case $\delta\mu_s$, $\delta\mu_{s'}$, $\delta\mu_{s''}$ are different, being the roots of the cubic equation (26), which we may write

$$\left.\begin{aligned} \delta\mu^3 - \frac{\delta\mu}{\nu_s^4}\left(\frac{a_{s'}a_{s''}}{\gamma_{s's''}^2} + \frac{a_s a_{s''}}{\gamma_{ss''}^2} + \frac{a_s a_{s'}}{\gamma_{ss'}^2}\right) + \frac{2}{\nu_s^6}\,\frac{a_s a_{s'} a_{s''}}{\gamma_{s's''}\gamma_{ss''}\gamma_{s's}}& \\ - \frac{\mu_s}{\nu_s^4}\left(\frac{a_{s'}a_{s''}}{\gamma_{s's}^2} + \frac{a_s a_{s''}}{\gamma_{ss''}^2} + \frac{a_s a_{s'}}{\gamma_{ss'}^2}\right) = 0& \end{aligned}\right\} . \tag{56}$$

Thus n_s, $n_{s'}$ and $n_{s''}$ will differ, and thus the mean value of

$$\sin(n_s t + \alpha_s)\,\sin(n_{s'}t + \alpha_{s'})$$

is zero. From equations (39) to (41) we find

$$\begin{aligned} \dot\phi_s\dot\phi_{s'} = {} & \frac{n_s^2\varkappa_s' C_s^2}{2} + \frac{n_{s'}^2\varkappa_{s'} C_{s'}^2}{2} + \frac{n_{s''}^2\varkappa_{s''}\varkappa_{s''}' C_{s''}^2}{2} \\ & - \sum_r \frac{\nu_r^2 n_r^2}{\nu_r^2 - \nu_s^2}\,\frac{a_s}{\gamma_{sr}}\,\lambda_{s'r}'\,\frac{C_r^2}{2} \\ & - \sum_{r'} \frac{\nu_{r'}^2 n_{r'}^2}{\nu_{r'}^2 - \nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{s'r'}}\,\lambda_{sr'}\,\frac{C_{r'}^2}{2} \\ & + \sum_{r''} n_{r''}^2 \lambda_{sr''}\lambda_{s'r''}'\,\frac{C_{r''}^2}{2}, \\ \dot\phi_q\dot\phi_{s'} = {} & -\frac{n_s^2\nu_s^2}{\nu_s^2 - \nu_q^2}\,\frac{a_q}{\gamma_{sq}}\,\varkappa_s'\,\frac{C_s^2}{2} - \frac{n_{q'}^2\nu_{q'}^2}{\nu_{q'}^2 - \nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{s'q'}}\,\varkappa_{q'}\,\frac{C_{q'}^2}{2} \\ & + n_{s'}^2\lambda_{qs'}\,\frac{C_{s'}^2}{2} + n_q^2\lambda_{s'q}'\,\frac{C_q^2}{2} + n_{s''}^2\varkappa_{s''}'\lambda_{qs''}\,\frac{C_{s''}^2}{2} + n_{q''}^2\varkappa_{q''}\lambda_{s'q''}'\,\frac{C_{s''}^2}{2} \\ & - \sum_{r'} \frac{\nu_{r'}^2 n_r^2}{\nu_{r'}^2 - \nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{s'r'}}\,\lambda_{qr'}\,\frac{C_{r'}^2}{2} - \sum_r \frac{\nu_r^2 n_r^2}{\nu_r^2 - \nu_q^2}\,\frac{a_q}{\gamma_{qr}}\,\lambda_{s'r}'\,\frac{C_r^2}{2} \\ & + \sum_{r''} n_{r''}^2\lambda_{s'r''}'\lambda_{qr''}\,\frac{C_{r''}^2}{2}. \end{aligned}$$

In all cases the summations are taken so as to exclude infinite terms. Substituting the value of the λ's from p. 51, we may rewrite these expressions,

$$\left.\begin{aligned} \dot\phi_s\dot\phi_{s'} = {} & \frac{n_s^2\varkappa_s' C_s^2}{2} + \frac{n_{s'}^2\varkappa_{s'} C_{s'}^2}{2} + \frac{n_{s''}^2\varkappa_{s''}\varkappa_{s''}' C_{s''}^2}{2} \\ & + \sum_{r,r'} \frac{n_r^2\nu_r^4}{(\nu_r^2 - \nu_s^2)^2}\,\frac{a_s}{\gamma_{sr}}\,\frac{a_{s'}}{\gamma_{s'r'}}\,\varkappa_r'\,\frac{C_r^2}{2} \\ & + \sum_{r',r} \frac{n_{r'}^2\nu_{r'}^4}{(\nu_r^2 - \nu_{s'}^2)^2}\,\frac{a_s}{\gamma_{sr}}\,\frac{a_{s'}}{\gamma_{s'r'}}\,\varkappa_{r'}\,\frac{C_{r'}^2}{2} \\ & - \sum_{r,r'r''} \frac{n_{r''}^2\nu_r^4}{(\nu_{r''}^2 - \nu_s^2)(\nu_{r''}^2 - \nu_{s'}^2)}\,\frac{a_s}{\gamma_{rs}}\,\frac{a_{s'}}{\gamma_{r's'}}\,\varkappa_{r''}\varkappa_{r''}''\,\frac{C_{r''}^2}{2} \end{aligned}\right\} . \tag{57}$$

$$\left.\begin{aligned}\dot{\phi}_q\dot{\phi}_{s'} = &-\frac{n_s^2\nu_s^2}{\nu_s^2-\nu_q^2}\,\frac{a_q}{\gamma_{sq}}\,\varkappa_s'\,\frac{C_s^2}{2}-\frac{n_{q'}^2\nu_{q'}^2}{\nu_{q'}^2-\nu_{s'}^2}\,\frac{a_{s'}}{\gamma_{s'q'}}\,\varkappa_{q'}\,\frac{C_{q'}^2}{2}\\ &-\frac{n_{s'}^2\nu_{s'}^2}{\nu_{s'}^2-\nu_{q'}^2}\,\frac{a_q}{\gamma_{sq}}\,\varkappa_{s'}\,\frac{C_{s'}^2}{2}-\frac{n_q^2\nu_q^2}{\nu_q^2-\nu_s^2}\,\frac{a_{s'}}{\gamma_{s'q'}}\,\varkappa_q'\,\frac{C_q^2}{2}\\ &-\left\{\frac{n_{s''}^2\nu_{s''}^2}{\nu_{s''}^2-\nu_{q''}^2}\,\frac{a_q}{\gamma_{sq}}\,\varkappa_{s''}'\varkappa_{s''}+\frac{n_{q''}^2\nu_{q''}^2}{\nu_{q''}^2-\nu_{s''}^2}\,\frac{a_{s'}}{\gamma_{s'q'}}\,\varkappa_{q''}'\varkappa_{q''}\right\}\frac{C_{s''}^2}{2}\\ &+\sum_{r,\,r'}\frac{\nu_r^4 n_r^2}{(\nu_r^2-\nu_s^2)(\nu_r^2-\nu_q^2)}\,\frac{a_q}{\gamma_{qr}}\,\frac{a_{s'}}{\gamma_{s'r'}}\,\varkappa_{r'}\,\frac{C_{r'}^2}{2}\\ &+\sum_{r',\,r}\frac{\nu_r^4 n_r^2}{(\nu_r^2-\nu_q^2)(\nu_r^2-\nu_s^2)}\,\frac{a_q}{\gamma_{rs}}\,\frac{a_{s'}}{\gamma_{r's'}}\,\varkappa_r'\,\frac{C_r^2}{2}\\ &+\sum_{r'',\,r'}\frac{n_{r''}^2\nu_{r''}^4}{(\nu_{r''}^2-\nu_{s'}^2)(\nu_{r''}^2-\nu_q^2)}\,\frac{a_s}{\gamma_{qs}}\,\frac{a_{s'}}{\gamma_{r's'}}\,\varkappa_{r''}'\varkappa_{r''}\,\frac{C_{r''}^2}{2}\end{aligned}\right\}. \tag{58}$$

Now the $\varkappa$'s are of zero order in $[1/a]$. Hence, if we want to find the value of $\dot{\phi}_s\dot{\phi}_{s'}$ and $\dot{\phi}_q\dot{\phi}_{s'}$ to the order $[1/a]^2$, we must substitute for n_s^2,

$$n_s^2 = \nu_s^2\{1+\nu_s^2(\mu_s+\delta\mu_s)\}$$

in the three first terms of $\dot{\phi}_s\dot{\phi}_{s'}$. Further, in the same terms, when they occur in the force function, we must substitute the value of $\gamma_{ss'}$ given in terms of $\gamma_{gg'}$ on p. 58, thus introducing the *aspect influence*. Owing to the same arguments as I have before used, these terms in the aspect influence containing first powers of cosines (see p. 58) will vanish for an *amorphic* body, when we are considering the law of cohesive force and not the force between any given individual pair of molecules. Thus the aspect influence would not be sensible below terms of the order $[1/a]^6$, which it must, however, be remarked is the order of terms which the second, third and fourth lines of $\dot{\phi}_s\dot{\phi}_{s'}$ contribute to the force.

As we cannot express analytically the solution of the cubic (56), we cannot substitute the value of n_s and $\varkappa_s'$, etc., in terms of $\gamma_{s''s'}$, $\gamma_{s''s}$ and $\gamma_{ss'}$; thus it is impossible to reduce the value of $\dot{\phi}_s\dot{\phi}_{s'}$ to terms in $\gamma_{g'g''}$, $\gamma_{gg''}$ and $\gamma_{g'g}$, and so show the exact nature of the aspect influence. But we may note that the aspect influence when $[1/m]$ is of the order $[1/a]^2$ plays a much larger part in the inter-molecular force than previously. Hence if the nearness of the molecules of a solid is of this order we should expect 'aspect' as well as 'modified action' to contribute towards the probability of multi-constancy.

The whole of the reasoning for this case is independent of the equality or

inequality of the phases. Thus, so far as the hypotheses of 'aspect' and 'modified action' are concerned, we might conclude from the present theory that—

(i). Neither has sensible influence if $[1/m]$ is not comparable with $[1/a]^3$.
(ii). The latter only has " " is comparable with $[1/a]^3$.
(iii). Both have " " " " $[1/a]^2$.

In the third case, however, only the latter action influences the terms of the very first importance in inter-molecular force.

Returning now to the formulae (57) and (58), suppose we retain only the terms to order $[1/a]^3$ in the force function, neglecting also those aspect influence terms which, although of this order, would in the case of an amorphic solid disappear for the mean of a great number of molecules. We find

$$\left.\begin{aligned} U = & \frac{1}{4\pi\rho}\Sigma\frac{\dot{\phi}_s\dot{\phi}_{s'}}{\gamma_{ss'}} + \frac{1}{4\pi\rho}\Sigma\frac{\dot{\phi}_q\dot{\phi}_{s'}}{\gamma_{qs'}} + \frac{1}{4\pi\rho}\Sigma\frac{\dot{\phi}_s\dot{\phi}_{q'}}{\gamma_{sq'}} \\ = & \frac{1}{8\pi\rho}\,\frac{1}{\gamma_{gg'}}\sum_{s,s's''}\{\nu_s^2\varkappa_s'C_s^2 + \nu_{s'}^2\varkappa_{s'}C_{s'}^2 + \nu_{s''}^2\varkappa_{s''}\varkappa_{s''}'C_{s''}^2\} \\ & - \frac{1}{8\pi\rho}\,\frac{1}{\gamma_{gg'}}\sum_s\sum_q\frac{\nu_s^4}{\nu_s^2-\nu_q^2}\cdot\frac{a_q}{\gamma_{sq}}(\varkappa_s'C_s^2 + \varkappa_{s'}C_{s'}^2) \\ & - \frac{1}{8\pi\rho}\,\frac{1}{\gamma_{gg'}}\sum_s\sum_q\frac{\nu_q^4}{\nu_q^2-\nu_s^2}\,\frac{a_{s'}}{\gamma_{s'q'}}(\varkappa_{q'}C_{q'}^2 + \varkappa_q'C_q^2) \\ & - \frac{1}{8\pi\rho}\,\frac{1}{\gamma_{gg'}}\sum_s\sum_q\left(\frac{\nu_s^4}{\nu_s^2-\nu_q^2}\,\frac{a_q}{\gamma_{sq}}\varkappa_{s''}'\varkappa_{s''} + \frac{\nu_q^4}{\nu_q^2-\nu_s^2}\,\frac{a_{s'}}{\gamma_{s'q'}}\varkappa_{q''}'\varkappa_{q''}\right)C_{s''}^2 \end{aligned}\right\} \quad (59)$$

Here the first sum is to take every possible value of s with the corresponding values of s' and s'', while the last three double sums are with regard to all values of s and q, except $s = q$.

We cannot express the inter-molecular force as some simple inverse power of inter-molecular distances, as the $\varkappa$'s are complicated functions of those distances involving the modifying action. We are still able, however, to deduce some general results from the above value of U.

a). The force between two molecules (when $[1/m]$ is of the order $[1/a]^2$, depends, so far as its chief terms are concerned, on the influence of kin-atoms) either in that which they exert directly on each other or on that indirect influence which they have in forcing pulsations of their own mutually modified periods upon their colleagues in their respective groups.

b). If we examine the values of the $\varkappa$'s given Art. 16, we see that if $[1/m]$ be not of order $[1/a]^2$, they are negative, as the second terms in both numerator

and denominator are small as compared with the first. If the second term in either grows in importance, x will still remain negative so long as our approximation to its value remains true, or otherwise x would pass through either zero or infinity, neither of which is admissible. Thus the first two terms of the first line of U will remain negative. Hence the force will be repulsive unless the third term summed for every modifying molecule rises to a greater value than the first two. Now x will I think always be necessarily less than unity, hence the third term of the order x^3 is less than the first two in order of magnitude. It may or may not, however, rise owing to the summation for all modifying molecules. The possibility, I should prefer to say probability, of the intermolecular force becoming negative for $[1/m] = [1/a]^2$ is, however, sufficiently clear.

22). If the above discussion on cohesion has seemed to a certain extent inclusive, this is only partially due to the complexity of the formulae, which, under any theory of cohesion, are bound to be complex if they are to explain the enormous variety of physical phenomena which we may include under that head. The real difficulty of the discussion lies rather in our ignorance as to whether—

α). Like atoms in different molecules have the same or nearly the same phase or not.

β). Molecular distance is comparable with atomic, and if so, what is the exact degree of the comparison.

The discussion, however, seems to have brought out a good deal that remained obscure in the treatment of cohesion in my first paper. It has, I think, shown that the ether squirt theory—

α). Leads to multi-constant equations of elasticity as a result rather of the truth of the 'hypothesis of modified action' than of that of 'aspect.'

β). Suggests that it is to the mutual influence of kin-atoms in different molecules that we must look for explanation of many of the facts of cohesion as well as for the origin of bands and other phenomena of spectrum analysis.

γ). Accounts for a change in sign as well as for a total change in character of the law of inter-molecular force as we vary the mean inter-molecular distance from something incomparably greater than inter-atomic distance down to distances of the order $[1/a]^2$.

δ). Enables us finally to lay aside for heavy matter all notion of 'force or

action at a distance.' All such force is only apparent, arising from the terms in the kinetic energy of the whole system due to the insensible motion of the ether.

23). It must be remembered that the conclusions of this paper do not replace but supplement the results obtained in the three previous papers. The laws of chemical action and the theory of the spectrum given in the first paper, the theory of optic and magnetic phenomena developed in the second paper, and lastly, the equations of generalized elasticity as discussed in the third paper, all flow as readily from the ether squirt as from the pulsating spherical atom. The former atom is, therefore, even better suited than the latter to be used as a model dynamical system for deducing the differential equations which express the laws of physical phenomena. It is, some may think, unlikely that the molecule is really a group of ether squirts, but the molecule is a dynamical system, and any model of a molecule which does not contradict obvious physical facts, but goes a long way to explain those facts, cannot but be suggestive as to the nature of the laws governing real molecular systems. The modifying action of kin-atoms in like molecules seems a suggestion of this kind sufficiently valuable to make it worth while to study closely the ether squirt hypothesis.

The application of this hypothesis to the generalized equations of elasticity and to the laws connecting pressure and volume in a gas must be deferred for the present, considering the great length to which this paper has already run.

On the Matrix which Represents a Vector.

By C. H. Chapman.

The fundamental idea of this work is emphasized by Buchheim as follows: "It is, or ought to be, well known that the linear and vector function of a vector is simply the matrix of the third order,"* meaning, of course, that the matrix is the operator which transforms ρ into $\phi\rho$ (Tait's Quaternions, p. 98. I shall refer to Tait's work from time to time simply as "Tait," the edition being that of 1867). It is merely a very special application of this remark to observe that any vector whatever whose rectangular components have the lengths x, y, z may be derived from that of which the components are of length 1, 1, 1 by the operation of the matrix

$$\left(\begin{array}{ccc} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{array}\right|,$$

and it follows that the study of vectors may be made to depend upon that of these matrices. The object of the present paper is to carry out and illustrate this study in some detail.

In the course of the work the matrix is freely spoken of as the vector by an allowable looseness of language, since no confusion can result; but it is not assumed that they are the same—in fact I wish to guard most carefully against that assumption.

The algebra of these matrices is the same as that of scalar quantities, inasmuch as, owing to their very special form, they are commutative with one another in multiplication; for this reason I have found it necessary to employ only a few very elementary properties of matrices. In fact Cayley's celebrated memoir (Phil. Trans. 1858) contains nearly all the theorems which are here

* Proc. London Math. Soc., Vol. XVI, p. 63; Clifford, Dynamic, p. 186.

made use of, although I have referred by preference to the papers by Sylvester (*Am. Journ. Math.*, Vol. VI) and Taber (*Am. Journ. Math.*, Vol. XII) as being more recently in the hands of all readers. The learned article of Taber contains a statement of pretty nearly the entire theory of matrices, at least from his point of view. The only new symbol here introduced is that of a circular substitution, *cy* or *cy'*, which has proved very interesting and useful; for the rest, the symbols S and V, not as selective but merely appellative, but with their geometrical significance essentially unchanged, have been found sufficient.

Many, if not all, the advantages which result from the use of quaternions will be found attained or attainable in this paper without their aid by the use of these matrices; the equations and formulas are sometimes identical in appearance with those of the quaternion calculus, and the resemblance has been systematically cherished—partly to aid in making comparisons, partly on account of the extreme elegance attained by the quaternion calculus in the hands of Hamilton and Tait.

The unpublished Vector Analysis of J. Willard Gibbs is briefly referred to in this article as V. A. G.

1.—*Properties of the Matrix.*

The matrix $\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} = \rho$ is referred to its axes,* which will be assumed as a rectangular system and, for distinction, designated as the axes of i, j and k respectively. The matrix ρ denotes that substitution which transforms the vector whose components along the three axes have the lengths 1, 1, 1 into that whose components are of lengths x, y, z respectively. The matrix therefore defines without ambiguity the length and direction of this vector, and I shall say that the matrix is the analytical expression or representation of the vector.

In particular the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ represents a vector whose length is $\sqrt{3}$ and which makes equal angles with the positive directions of the axes. In the

* Taber, *Am. Journ. Math.*, Vol. XII, p. 360.

vector theory this matrix plays the part of unity in the scalar and will be spoken of as vector unity and denoted by v.*

For brevity, matrices of the kind here considered will be denoted by Greek letters, and the same letter will uniformly denote both the matrix and the vector represented by it. If

$$\alpha = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} \text{ and } \beta = \begin{vmatrix} a' & 0 & 0 \\ 0 & b' & 0 \\ 0 & 0 & c' \end{vmatrix},$$

then

$$\alpha + \beta = \begin{vmatrix} a + a', & 0\ , & 0 \\ 0\ , & b + b', & 0 \\ 0\ , & 0\ , & c + c' \end{vmatrix}.\dagger$$

Whence it is evident that the sum of the matrices represents the sum of the vectors.‡

The product of the matrices α and β is

$$\alpha\beta = \begin{vmatrix} aa' & 0 & 0 \\ 0 & bb' & 0 \\ 0 & 0 & cc' \end{vmatrix},$$

which is evidently the same as $\beta\alpha$, and, for the purposes of this paper, the product of the vectors α and β will be so defined that it may be represented by the matrix $\alpha\beta$; that is, this product is a new vector whose component lengths are the products of those of the factors. The writer is well aware of his presumption in thus departing from illustrious usage,§ but inasmuch as he thus attains analytical consistency, simplicity of notation, and preservation of the associative law, both of which latter are lacking in the ingenious work of J. Willard Gibbs, and at the same time retains the freedom of commutative multiplication, the lack of which makes Hamilton's system at once so significant, so fascinating, and so difficult for ordinary students to master and apply, he has ventured, in a tentative way, to make the departure. Unless all the constituents of the matrix except those of the principal diagonal are zeros, $\alpha\beta$ is not in general the same as $\beta\alpha$, so that I by no means imply by this that linear vector functions are in general commutative in multiplication.

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* Sylvester, *Am. Journ. Math.*, Vol. VI, pp. 274 and 275.

† Sylvester, *Am. Journ. Math.*, Vol. VI, pp. 274 and 275.

‡ V. A. G., p. 4.

§ Tait, p. 44 ff.; V. A. G., p. 5.

Since division of matrices is defined by the equation

$$\beta\frac{\alpha}{\beta}=\alpha,^*$$

we see at once that the quotient of the matrix α by β is the matrix

$$\begin{pmatrix} \frac{a}{a'} & 0 & 0 \\ 0 & \frac{b}{b'} & 0 \\ 0 & 0 & \frac{c}{c'} \end{pmatrix}$$

which will be taken to represent the quotient of the vector α by the vector β. From this it follows that the division is not ambiguous and that

$$\beta\frac{\alpha}{\beta}=\frac{\alpha}{\beta}\beta=\alpha.$$

It is to be observed that this division is indeterminate or gives a quotient with an infinite constituent in cases quite analogous to those of scalar division.†

Without using Sylvester's Theorem,‡ it is evident from the rules for multiplication and division that any function $f(\rho)$ which can be expanded in positive or negative powers of ρ will be denoted by the matrix

$$\begin{pmatrix} f(x) & 0 & 0 \\ 0 & f(y) & 0 \\ 0 & 0 & f(z) \end{pmatrix}.$$

The matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

will be denoted by i, j, k respectively. For their multiplication we have

$$i^2=i,\ \ j^2=j,\ \ k^2=k,\ \ ij=ik=jk=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}=ijk.$$

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* Sylvester, *Am. Journ. Math.*, Vol. VI, p. 276.

† V. A. G., p. 47, Art. 128; Sylvester, *Am. Journ. Math.*, Vol. VI, p. 274.

‡ Quoted and verified by Taber, p. 378; Buchheim, Proc. Lond. Math. Soc., Vol. 16, p. 81.

Also $\quad i\rho = xi;\ j\rho = yj;\ k\rho = zk;$

whence $\quad (i+j+k)\rho = v\rho = \rho = i\rho + j\rho + k\rho;$

being a formula for decomposing a vector into three rectangular components.

The matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I, J, K,$$

have the properties that

$$\begin{aligned} &I^2 = I,\ J^2 = J,\ K^2 = K, \\ &IJ = JI = j;\ IK = i;\ JK = k, \\ &iI = i;\ iJ = 0;\ iK = i, \\ &jI = j;\ jJ = j;\ jK = 0, \\ &kI = 0;\ kJ = k;\ kK = k. \end{aligned}$$

The vectors $i, j, k;\ I, J, K$ cannot be used as divisors.

2.—*The Operators cy and cy′ and the Symbols S and V.*

If ρ denote the matrix $\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$, or briefly (x, y, z), then shall $cy\rho$ denote (y, z, x) and $cy^2\rho$, (z, x, y); cy being an abbreviation for the word cyclic and denoting evidently a circular substitution among the constituents of ρ.*
Clearly also

$$cy^3\rho = \rho.$$

Again, if α, β, γ are any three rectangular unit vectors and $\rho = x\alpha + y\beta + z\gamma$, cy' shall denote the substitution

$$(\alpha, \beta, \gamma;\ \gamma, \alpha, \beta),$$

so that $\quad cy'\rho = x\gamma + y\alpha + z\beta;\ cy'^2\rho = x\beta + y\gamma + z\alpha,\ cy'^3\rho = \rho.$

Denoting $\alpha + \beta + \gamma$ by v', cy and cy' are operators which rotate ρ in a negative direction about the axes v and v' respectively, through an angle equal to $\frac{2\pi}{3}$, while the rotation performed by cy^2 and cy'^2 is $\frac{4\pi}{3}$.

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* Cf. Taber, *Am. Journ. Math.*, Vol. XIII, p. 163.

If $\alpha = (a, b, c)$ and $\beta = (a', b', c')$ are any two matrices of the kind in question, then $\alpha\beta = (aa', bb', cc')$ and

$$S.\alpha\beta = aa' + bb' + cc'$$

will be called the scalar of $\alpha\beta$. It is the product of the lengths of the vectors into the cosine of their included angle, and has thus its usual geometrical meaning. Its practical use is the same as in quaternions. Since in any product the vector v may be suppressed, the expression $S.v\rho = x + y + z$ will often be written simply $\qquad S.\rho.$

Obviously $\qquad cyS\rho = Scy\rho$ and $cyS.\alpha\beta = Scy(\alpha\beta)$.

Also $\qquad cy(\alpha, \beta, \gamma, \delta \ldots .) = cy\alpha\, cy\beta\, cy\gamma \ldots .$

and $\qquad cy\dfrac{\alpha}{\beta} = \dfrac{cy\alpha}{cy\beta}.$

While $\qquad cy(\alpha cy\beta) = cy\alpha cy^2\beta$ and $S.\alpha cy^2\beta = S.\beta cy\alpha$,

from the definitions of the symbols. We recall also that if $S.\alpha\beta = 0$, α and β are perpendicular to each other, and note that the product of any two vectors perpendicular to each other is a third vector perpendicular to vector unity.

In particular,

$$cyi = k\ ,\quad cyk = j\,,\quad cyj = i,$$
$$cyI = K,\quad cyK = J,\quad cyJ = I.$$

Let us now form the vector

$$\delta = cy\alpha\, cy^2\beta - cy\beta\, cy^2\alpha.$$

We have by the above formulas

$$S.\alpha\,(cy\alpha\, cy^2\beta - cy\beta\, cy^2\alpha) = S.\alpha cy\alpha . cy^2\beta - S.\alpha cy\alpha\, cy^2\beta = 0;$$

and likewise $\qquad S.\beta\,(cy\alpha\, cy^2\beta - cy\beta\, cy^2\alpha) = 0.$

Hence δ is perpendicular to α and β. Its direction is such that a positive rotation about it as an axis turns α to β, and its length is the product of the lengths of α and β into the sine of their included angle; it is therefore the vector which Hamilton denotes by $V\alpha\beta$; but since we do not derive it from α and β by multiplication, it will be denoted in this paper by $V(\alpha, \beta)$, or $-V(\beta, \alpha)$.

We have $\qquad V(i, j) = k;\ \ V(j, k) = i,\ \ V(k, i) = j.$

Also, if α, β, γ are any rectangular system of unit vectors,

$$V(\alpha,\beta)=\gamma;\ V(\gamma,\alpha)=\beta;\ V(\beta,\gamma)=\alpha.$$

We observe now that taking $\alpha+\beta+\gamma=v'$ for the axis of cy', then

$$cy'\alpha=\gamma,\ cy'\beta=\alpha,\ cy'\gamma=\beta;$$

and that if $\rho=x\alpha+y\beta+z\gamma$, then

$$S.v'\rho=x+y+z.$$

Finally, however ρ may be expressed, the square of its length is

$$S.v\rho^2=S.\rho^2.$$

It will now be clear that

$$S.\alpha V(\beta,\gamma)=S.\beta V(\gamma,\alpha)=S.\gamma V(\alpha,\beta),\text{*}$$

whatever vectors α, β and γ may be; and that if $S.\alpha V(\beta,\gamma)=0$, the three vectors lie in the same plane.

The coefficients x, y and z in the expression

$$\delta=x\alpha+y\beta+z\gamma,$$

where α, β, γ do not lie in the same plane, can now be determined. Multiplying by $V(\alpha,\beta)$ and taking the scalars, we find

$$z=\frac{S.\delta V(\alpha,\beta)}{S.\gamma V(\alpha,\beta)} \tag{9}$$

and similarly

$$x=\frac{S.\delta V(\beta,\gamma)}{S.\alpha V(\beta,\gamma)};\ y=\frac{S.\delta V(\gamma,\alpha)}{S.\beta V(\gamma,\alpha)}.$$

Whence

$$\delta S.\alpha V(\beta,\gamma)=\alpha S.\delta V(\beta,\gamma)+\beta S.\delta V(\gamma,\alpha)+\gamma S.\delta V(\alpha,\beta).\dagger \tag{1}$$

This expression is scarcely less simple than the corresponding one in quaternions, from which it differs in form only by the notation $V(\alpha,\beta)$ instead of $V.\alpha\beta$. In the quaternion formula $S.(\alpha V.\beta\gamma)$ the V may be omitted without changing the meaning, but from $S.\alpha V(\beta,\gamma)$ it cannot.

It is also possible to expand δ in the form

$$\delta=xV(\gamma,\alpha)+yV(\alpha,\beta)+zV(\beta,\gamma).$$

* Cf. Tait, pp. 56 and 65. † Tait, p. 57.

We have

$$z = \frac{S.\alpha\delta}{S.\alpha V(\beta,\gamma)};\quad y = \frac{S.\gamma\delta}{S.\gamma V(\alpha,\beta)};\quad x = \frac{S.\beta\delta}{S.\beta V(\gamma,\alpha)}.$$

Hence

$$\delta S.\alpha V(\beta,\gamma) = V(\gamma,\alpha)\,S.\beta\delta + V(\alpha,\beta)\,S.\gamma\delta + V(\beta,\gamma)\,S.\alpha\delta. \qquad (2)$$

The following formulas which follow directly from the definitions of the symbols cy and S are useful in what follows:

$$\left.\begin{aligned} vS.\rho &= \rho + cy\rho + cy^2\rho, \\ vS.\rho^2 &= \rho^2 + cy\rho^2 + cy^2\rho^2, \\ \text{whence}\qquad v(S^2.\rho - S.\rho^2) &= 2vS.\rho cy\rho. \end{aligned}\right\} \qquad (3)$$

Also

$$v'S.v'\rho = \rho + cy'\rho + cy'^2\rho. \qquad (4)$$

The cosine of the angle which ρ makes with v is

$$\frac{S.\rho}{\sqrt{3S.\rho^2}},$$

and the cosine of the angle between ρ and v' is

$$\frac{S.v'\rho}{\sqrt{3S.\rho^2}}.$$

The components of ρ parallel and perpendicular to v are respectively

$$\tfrac{1}{3}vS.\rho \text{ and } \rho - \tfrac{1}{3}v.S\rho; \qquad (5)$$

while the same with respect to v' are

$$\tfrac{1}{3}v'S.v'\rho \text{ and } \rho - \tfrac{1}{3}v'S.v'\rho. \qquad (6)$$

3.—*The Symbols* $cy^{\frac{3\vartheta}{2\pi}}$ *and* $cy'^{\frac{3\vartheta}{2\pi}}$.

Since cy^3 and cy'^3 perform respectively negative rotations equal to 2π about their respective axes, it is very natural to indicate corresponding rotations through the angle ϑ by $cy^{\frac{3\vartheta}{2\pi}}$ and $cy'^{\frac{3\vartheta}{2\pi}}$. When $\vartheta = \frac{2\pi}{3}$,

$$cy^{\frac{3\vartheta}{2\pi}} = cy,\quad cy'^{\frac{3\vartheta}{2\pi}} = cy';$$

when $\vartheta = \frac{4\pi}{3}$,

$$cy^{\frac{3\vartheta}{2\pi}} = cy^2;\quad cy'^{\frac{3\vartheta}{2\pi}} = cy'^2.$$

I propose to find a matrix μ such that

$$\mu\rho = cy^{\frac{3\vartheta}{2\pi}}\rho.$$

Evidently for $\vartheta = \frac{2\pi}{3}$,

$$\mu = \begin{pmatrix} \frac{y}{x} & 0 & 0 \\ 0 & \frac{z}{y} & 0 \\ 0 & 0 & \frac{x}{z} \end{pmatrix} \tag{7}$$

and for $\vartheta = \frac{4\pi}{3}$,

$$\mu = \begin{pmatrix} \frac{z}{x} & 0 & 0 \\ 0 & \frac{x}{y} & 0 \\ 0 & 0 & \frac{y}{z} \end{pmatrix}. \tag{8}$$

We observe that ρ and $\mu\rho$ have the same length, and make the same angles with v; hence we obtain the equations

$$S.\mu\rho = S.\rho\,; \tag{9}$$
$$S.\mu^2\rho^2 = S.\rho^2. \tag{10}$$

To obtain the third necessary scalar equation, we note that ϑ is the angle between $\frac{1}{3}(3\rho - vS.\rho)$ and $\frac{1}{3}(3\mu\rho - vS.\mu\rho)$, the vector projections of ρ and $\mu\rho$ respectively perpendicular to v. Hence

$$\cos\vartheta = \frac{\frac{1}{3}S.(3\rho - vS.\rho)(3\mu\rho - vS.\mu\rho)}{\frac{1}{3}S.(3\rho - vS.\rho)^2} = \frac{3S.\mu\rho^2 - S.\rho S.\mu\rho}{3S.\rho^2 - S^2.\rho}.$$

Whence

$$3S.\mu\rho^2 = (3S.\rho^2 - S^2.\rho)\cos\vartheta + S^2.\rho\,. \tag{11}$$

Instead of solving these three equations directly for μ, let us approach the result indirectly by aid of a vector ν which rotates $\mu\rho$ through an angle ϕ, leading to the equations

$$S.\nu\mu\rho = S.\mu\rho = S.\rho, \tag{12}$$
$$S.\nu^2\mu^2\rho^2 = S.\mu^2\rho^2 = S.\rho^2, \tag{13}$$
$$3S.\nu\mu^2\rho^2 = (3S.\rho^2 - S^2.\rho)\cos\phi + S^2.\rho. \tag{14}$$

Again we notice that $\nu\mu$ is a vector which rotates ρ about v through the angle $\vartheta + \phi$ without altering its length, and thus deduce the additional equations

$$3S.\nu\mu\rho^2 = (3S.\rho^2 - S^2.\rho)\cos(\vartheta + \phi) + S^2\rho. \tag{15}$$

Now in general the vectors $\mu\rho$, $\mu\rho^2$, and $\mu^2\rho^2$ do not lie in the same plane, for we have

$$vS.\mu^2\rho^2 V(\mu\rho, \mu\rho^2) = vS.\mu^2 cy\mu cy^2\mu\rho^2 cy\rho cy^2\rho (cy^2\rho - cy\rho), \tag{16}$$

remembering that $cy\rho^2 = (cy\rho)^2$ and $cy^2\rho^2 = (cy^2\rho)^2$. And if we take into account the fact that

$$vS.\rho cy\rho cy^2\rho\sigma = \rho cy\rho cy^2\rho S.\sigma \tag{17}$$

we may write equation (16)

$$vS.\mu^2\rho^2 V(\mu\rho, \mu\rho^2) = \mu cy\mu cy^2\mu\rho cy\rho cy^2\rho S.\mu\rho (cy^2\rho - cy\rho). \tag{18}$$

If $S.\mu\rho (cy^2\rho - cy\rho) = 0$, then μ satisfies a fourth scalar equation besides (9), (10) and (11). This can happen, for a given value of ρ, only in very special cases, and one such case is $\mu = v$.

In general, then, by aid of equation (2), we may write

$$\nu . S.\mu\rho V(\mu\rho^2, \mu^2\rho^2)$$
$$= V(\mu^2\rho^2, \mu\rho) S.\nu\mu\rho^2 + V(\mu\rho, \mu\rho^2) S.\nu\mu^2\rho^2 + V(\mu\rho^2, \mu^2\rho^2) S.\nu\mu\rho. \tag{19}$$

In simplifying the formula (18) it will be useful to note that

$$V(\delta\alpha, \delta\beta) = cy\delta cy^2\delta . V(\alpha, \beta). \tag{20}$$

We may see this to be true since

$$V(\delta\alpha, \delta\beta) = cy(\delta\alpha) cy^2(\delta\beta) - cy(\delta\beta) cy^2(\delta\alpha)$$
$$= cy\delta cy^2\delta (cy\alpha cy^2\beta - cy\beta cy^2\alpha) = cy\delta cy^2\delta . V(\alpha, \beta).$$

Using this formula, equation (19) becomes

$$vS.\mu\rho cy(\mu\rho^2) cy^2(\mu\rho^2) V(v, \mu) = cy(\mu\rho) cy^2(\mu\rho) V(\mu\rho, v) S.\nu\mu\rho^2$$
$$+ cy(\mu\rho) cy^2(\mu\rho) V(v, \rho) S.\nu\mu^2\rho^2 + cy(\mu\rho^2) cy^2(\mu\rho^2) V(v, \mu) S.\nu\mu\rho. \tag{21}$$

Using the formula (17) we see that this equation is divisible by the factor $cy\mu\rho cy^2\mu\rho$, giving for the quotient

$$\nu\mu\rho S.cy\rho cy^2\rho V(v, \mu)$$
$$= V(\mu\rho, v) S.\nu\mu\rho^2 + V(v, \rho) S.\nu\mu^2\rho^2 + cy\rho cy^2\rho V(v, \mu) S.\nu\mu\rho. \tag{22}$$

To complete the determination of ν we shall assign to μ a particular value such that knowing the effect of ν on $\mu\rho$, we can readily find the vector which rotates ρ through the same angle. The vector $\mu = \frac{cy\rho}{\rho}$ satisfies equations (9),

(10) and (11) but not (18), except in the special cases when ρ is the vector to the surface
$$xy + yz + zx = x^2 + y^2 + z^2.$$

We may therefore take $\mu = \frac{cy\rho}{\rho}$, and the corresponding value of ϑ is $\frac{2\pi}{3}$. With these values equation (22) becomes

$$3\nu cy\rho S.cy\rho cy^2\rho V\left(v, \frac{cy\rho}{\rho}\right) = V(cy\rho, v)\left[(3S.\rho^2 - S^2.\rho)\cos\left(\frac{2\pi}{3} + \phi\right) + S^2.\rho\right]$$
$$+ V(v, \rho)[(3S.\rho^2 - S^2.\rho)\cos\phi + S^2.\rho] + 3cy\rho cy^2\rho V\left(v, \frac{cy\rho}{\rho}\right)S.\rho. \quad (23)$$

Now $$cy\rho cy^2\rho V\left(v, \frac{cy\rho}{\rho}\right) = cy\rho cy^2\rho\left(\frac{cy^3\rho}{cy^2\rho} - \frac{cy^3\rho}{cy\rho}\right) = \rho cy\rho - cy^2\rho^2,$$

and $$S.(\rho cy\rho - cy^2\rho^2) = S.\rho cy\rho - S.\rho^2;$$

since $$S.cy^2\rho^2 = cy^2 S.\rho^2 = S.\rho^2.$$

And from equations (3),
$$S.\rho cy\rho = \tfrac{1}{2}(S^2.\rho - S.\rho^2).$$

Hence $$S.cy\rho cy^2\rho V\left(v, \frac{cy\rho}{\rho}\right) = \tfrac{1}{2}(S^2.\rho - 3S.\rho^2).$$

Again,
$$\begin{aligned}
&[V(cy\rho, v) + V(v, \rho)]\,S^2.\rho + 3cy\rho cy^2\rho V\left(v, \frac{cy\rho}{\rho}\right)S.\rho \\
&\quad = V(v, \rho - cy\rho)\,S^2.\rho + 3V(\rho, cy\rho)\,S.\rho \\
&\quad = [V(\rho + cy\rho + cy^2\rho, \rho - cy\rho) + 3V(\rho, cy\rho)]\,S.\rho \\
&\quad = [V(\rho, cy\rho) + V(cy^2\rho, \rho - cy\rho)]\,S.\rho = \tfrac{1}{2}v\,[S^2.\rho - 3S.\rho^2]\,S.\rho.
\end{aligned}$$

Dividing out the common factor $[S^2.\rho - 3S.\rho^2]$ from equation (23) we have left

$$\tfrac{3}{2}\nu cy\rho = -V(cy\rho, v)\cos\left(\frac{2\pi}{3} + \phi\right) - V(v, \rho)\cos\phi + \tfrac{1}{2}v\,S.\rho,$$

or finally,
$$3\nu cy\rho = 2V(v, cy\rho)\cos\left(\frac{2\pi}{3} + \phi\right) + 2V(\rho, v)\cos\phi + vS.\rho. \quad (24)$$

Changing $cy\rho$ to ρ and consequently ρ to $cy^2\rho$, and recalling that
$$S.\rho = S.cy\rho = S.cy^2\rho,$$

we have, if we replace ν by μ,

$$3\mu\rho = 2V(v, \rho)\cos\left(\frac{2\pi}{3}+\phi\right) + 2V(cy^2\rho, v)\cos\phi + vS.\rho. \tag{25}$$

Now, inasmuch as ν was a vector which rotated $cy\rho$ through the angle ϕ, it must be that μ has the same effect on ρ.

Equation (25) is homogeneous in the tensor of ρ and μ is therefore not a function of the length of ρ, but depends merely on its direction.

As a partial verification of the correctness of this formula, I remark that when $\phi = 0$ it becomes

$$3\mu\rho = 3\rho, \text{ or } \mu = v,$$

as it should.

Since (25) gives one and only one value of μ for each value of ϕ, no two being the same, as ϕ varies from 0 to 2π, $\mu\rho$ will perform a complete rotation about the axis v and return to its original position.

We may now write

$$cy^{\frac{3\phi}{2\pi}}\rho = \mu\rho, \tag{26}$$

where μ is a known vector.

In a similar manner we may compute the vector factor which will produce a given rotation about any given axis.

In fact, denoting any vector axis of length $\sqrt{3}$ by $\alpha+\beta+\gamma$, where α, β and γ are three rectangular unit vectors so situated that the given axis is the diagonal of a cube of which α, β, γ are the edges, any vector whatever may be denoted by

$$\rho = x\alpha + y\beta + z\gamma, \tag{27}$$

and the problem is to find a substitution μ which rotates ρ through an angle ϑ about v'.

The equations of condition are

$$S.\mu\rho v' = S.\rho v', \tag{28}$$

$$S.\mu^2\rho^2 = S.\rho^2, \tag{29}$$

$$S.\mu\rho^2 - \tfrac{1}{3}S^2.v\mu\rho = \cos\vartheta\,(S.\rho^2 - \tfrac{1}{3}S^2.v'\rho),$$

or

$$S.\mu\rho^2 = \tfrac{1}{3}\left[\cos\vartheta\,(3S.\rho^2 - S^2.v'\rho) + S^2.v'\rho\right]. \tag{30}$$

Equation (30) is obtained as follows:

The component of ρ parallel to v' being tv', we shall have $S.v'(\rho - tv') = 0$, whence

$$tS.v'^2 = S.v'\rho,$$

or

$$t = \tfrac{1}{3}S.v'\rho.$$

Hence the component of ρ perpendicular to v' is

$$\rho - \tfrac{1}{3} v' S.v'\rho, \tag{31}$$

and that of $\mu\rho$ is

$$\mu\rho - \tfrac{1}{3} v' S.v'\mu\rho. \tag{32}$$

The vectors (31) and (32) have the same length. Observing that $S.v'^2 = 3$, and that the scalar of their product divided by the square of the length of either gives the cosine of the angle between them, we have

$$\cos\vartheta = \frac{S.\mu\rho^2 - \frac{1}{3} S^2.v'\mu\rho}{S.\rho^2 - \frac{1}{3} S^2.v'\rho},$$

from which equation (30) follows at once.

Proceeding as before we shall obtain three equations corresponding to (9), (10) and (11) respectively,

$$S.\nu\mu\rho^2 = \tfrac{1}{3}\left[\cos(\vartheta + \phi)(3S.\rho^2 - S^2.v'\rho) + S^2.v'\rho\right], \tag{33}$$

$$S.\nu\mu^2\rho^2 = \tfrac{1}{3}\left[\cos\phi\,(3S.\rho^2 - S^2.v'\rho) + S^2.v'\rho\right], \tag{34}$$

$$S.v'\mu\nu\rho = S.v'\rho. \tag{35}$$

Using the formula (2) to expand ν in terms of $\mu\rho^2$, $\mu^2\rho^2$, and $v'\mu\rho$, we obtain the equation

$$\begin{aligned}\nu S.v'\mu\rho\, V(\mu\rho^2, \mu^2\rho^2)\\ = V(\mu^2\rho^2, \mu\rho v')\, S.\nu\mu\rho^2 + V(\mu\rho v', \mu\rho^2)\, S.\nu\mu^2\rho^2 + V(\mu\rho^2, \mu^2\rho^2)\, S.v'\nu\mu\rho,\end{aligned} \tag{36}$$

which, by aid of (13), becomes

$$\begin{aligned}\nu S.v'\mu\rho cy(\mu\rho^2)\, cy^2(\mu\rho^2)\, V(v, \mu) = cy(\mu\rho)\, cy^2(\mu\rho)\, V(\mu\rho, v')\, S.\nu\mu\rho^2\\ + cy(\mu\rho)\, cy^2(\mu\rho)\, V(v', \rho)\, S.\nu\mu^2\rho^2 + cy(\mu\rho^2)\, cy^2(\mu\rho^2)\, V(v, \mu)\, S.v'\nu\mu\rho.\end{aligned} \tag{37}$$

From this we may divide out the factor $cy(\mu\rho)\, cy^2(\mu\rho)$, leaving

$$\left.\begin{aligned}\nu\mu\rho S.v' cy\rho cy^2\rho\, V(v, \mu)\\ = V(\mu\rho, v')\, S.\nu\mu\rho^2 + V(v', \rho)\, S.\nu\mu^2\rho^2 + cy\rho cy^2\rho\, V(v, \mu)\, S.v'\nu\mu\rho\\ = \tfrac{1}{3} S^2.v'\rho.\left[V(\mu\rho, v') + V(v', \rho)\right]\\ + \tfrac{1}{3}(3S.\rho^2 - S^2.v'\rho).\left[V(\mu\rho, v).\cos(\vartheta + \phi) + V(v', \rho)\cos\phi\right]\\ + S.v'\rho.cy\rho cy^2\rho\, V(v, \mu)\end{aligned}\right\} \tag{38}$$

by aid of equations (33), (34) and (35).

Equation (38) may be written

$$\begin{aligned}\nu\mu\rho S.v' V(\rho, \mu\rho) = \tfrac{1}{3} S^2.v'\rho.V(v', \rho - \mu\rho)\\ + \tfrac{1}{3}(3S.\rho^2 - S^2.v'\rho)\left[V(\mu\rho, v')\cos(\vartheta + \phi) + V(v', \rho)\cos\phi\right] + V(\rho, \mu\rho)\, S.v'\rho.\end{aligned}$$

If in this we assign to μ the value $\frac{cy'\rho}{\rho}$ and to ϑ the corresponding value of $\frac{2\pi}{3}$, it becomes

$$\left.\begin{aligned}\nu cy'\rho S.v' V(\rho, cy'\rho) &= \tfrac{1}{3} S^2.v'\rho.V(v', \rho - cy'\rho)\\ &+ \tfrac{1}{3}(3S.\rho^2 - S^2.v'\rho)\Big[V(cy'\rho, v')\cos\Big(\frac{2\pi}{3}+\phi\Big) + V(v', \rho)\cos\phi\Big]\\ &+ V(\rho, cy'\rho)S.v'\rho.\end{aligned}\right\}\quad(39)$$

The vector ν turns $cy'\rho$ through the angle ϕ around the axis v'; hence changing $cy'\rho$ to ρ and therefore $cy'^2\rho$ to $cy'\rho$ and ρ to $cy'^2\rho$, we shall have the vector μ sought for:

$$\left.\begin{aligned}\mu\rho S.v' V(cy'^2\rho, \rho) &= \tfrac{1}{3} S^2.v'\rho.V(v', cy'^2\rho - \rho)\\ &+ \tfrac{1}{3}(3S.\rho^2 - S^2.v'\rho)\Big[V(\rho, v')\cos\Big(\frac{2\pi}{3}+\phi\Big) + V(v', cy'^2\rho)\cos\phi\Big]\\ &+ V(cy'^2\rho, \rho)S.v'\rho.\end{aligned}\right\}\quad(40)$$

Now $cy'^2\rho = x\beta + y\gamma + z\alpha$ and $V(cy'^2\rho, \rho)$

$$\begin{aligned}&= (xcy\beta + ycy\gamma + zcy\alpha)(xcy^2\alpha + ycy^2\beta + zcy^2\gamma)\\ &\qquad\qquad - (xcy^2\beta + ycy^2\gamma + zcy^2\alpha)(xcy\alpha + ycy\beta + zcy\alpha)\\ &= (yz - x^2)V(\alpha, \beta) + (zx - y^2)V(\beta, \gamma) + (xy - z^2)V(\gamma, \alpha)\\ &= (zx - y^2)\alpha + (xy - z^2)\beta + (yz - x^2)\gamma,\end{aligned}$$

owing to the property of three rectangular unit vectors that

$$V(\alpha, \beta) = \gamma;\quad V(\beta, \gamma) = \alpha;\quad V(\gamma, \alpha) = \beta.*$$

We conclude that

$$S.v' V(cy'^2\rho, \rho) = (zx + xy + yz) - (x^2 + y^2 + z^2) = -\tfrac{1}{2}(3S.\rho^2 - S^2.v'\rho).$$

Also

$$\begin{aligned}V(v', cy'^2\rho - \rho) &= V(v', (z-x)\alpha + (x-y)\beta + (y-z)\gamma)\\ &= (cy\alpha + cy\beta + cy\gamma)((z-x)cy^2\alpha + (x-y)cy^2\beta + (y-z)cy^2\gamma)\\ &\quad - (cy^2\alpha + cy^2\beta + cy^2\gamma)((z-x)cy\alpha + (x-y)cy\beta + (y-z)cy\gamma),\end{aligned}$$

which reduces to $(2y - x - z)\alpha + (2z - x - y)\beta + (2x - y - z)\gamma$. Hence

$$\tfrac{1}{3}S^2.v'\rho.V(v', cy'^2\rho - \rho) + S.v'\rho.V(cy'^2\rho, \rho) = -\frac{1}{6}(3S.\rho^2 - S^2.v'\rho)S.v'\rho.$$

Dividing out the common factor v', equation (36) reduces to

$$\tfrac{1}{2}\mu\rho = -\tfrac{1}{3}\Big[V(\rho, v')\cos\Big(\frac{2\pi}{3}+\phi\Big) + V(v', cy'^2\rho)\cos\phi\Big] + \frac{1}{6}v'S.v'\rho,$$

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*Cf. Tait's Quaternions, Art. 66.

or $$3\mu\rho = v' S.v'\rho - 2\left[V(\rho, v') \cos\left(\frac{2\pi}{3} + \phi\right) + V(v', cy'^2\rho) \cos\phi \right], \tag{41}$$

a formula which is very similar to (22), to which it reduces when $v' = v$.

When $\phi = 90°$ and ρ is perpendicular to v', equation (38) becomes

$$3\mu\rho = \sqrt{3}\, V(\rho, v'), \tag{42}$$

as it should.

Taking $S.v'\rho = 0$, we have now

$$3cy'^{\frac{3\vartheta}{2\pi}}\rho = V(v', \rho) \cos\left(\frac{2\pi}{3} + \vartheta\right) + V(cy'^2\rho, v') \cos\vartheta; \tag{43}$$

and differentiating this with respect to ϑ, we find that

$$\left.\begin{aligned} d.3cy'^{\frac{3\vartheta}{2\pi}}\rho &= -V(v', \rho) \sin\left(\frac{2\pi}{3} + \vartheta\right) d\vartheta - V(cy'^2\rho, v') \sin\vartheta d\vartheta \\ &= 3cy'^{\frac{3\left(\vartheta + \frac{\pi}{2}\right)}{2\pi}} \rho . d\vartheta . \end{aligned}\right\} \tag{44}$$

Writing for brevity $cy'^{\frac{3\vartheta}{2\pi}} = q^{\vartheta}$, so that $q\rho^{2\pi} = cy'^{3}\rho = \rho$, and observing that $V(v', \rho) = xq^{-\frac{\pi}{2}}\rho$, and $V(cy'^2\rho, v') = V(q^{\frac{4\pi}{3}}\rho, v') = -q^{-\frac{\pi}{2}}q^{\frac{4\pi}{3}}\rho = -q^{\frac{5\pi}{6}}\rho$ we can write equations (39) and (40) in the form

$$3q^{\vartheta}\rho = q^{-\frac{\pi}{2}}\rho \cos\left(\frac{2\pi}{3} + \vartheta\right) - q^{\frac{5\pi}{6}}\rho \cos\vartheta \tag{45}$$

and $$d.q^{\vartheta}\rho = q^{\vartheta + \frac{\pi}{2}} . \rho d\vartheta . \tag{46}$$

We now perceive that taking α a unit vector perpendicular to v', the equation of a plane curve may be written

$$\rho = rq^{\vartheta}\alpha \tag{47}$$

where r is a scalar. Differentiating twice in succession with respect to the time, we find that

$$\rho' = r'q^{\vartheta}\alpha + rq^{\vartheta + \frac{\pi}{2}}\alpha . \frac{d\vartheta}{dt}, \tag{48}$$

$$\rho'' = \left(r'' - r\left(\frac{d\vartheta}{dt}\right)^2\right) q^{\vartheta}\alpha + \left(2r'\frac{d\vartheta}{dt} + r\frac{d^2\vartheta}{dt^2}\right) q^{\vartheta + \frac{\pi}{2}}\alpha . \tag{49}$$

These equations give the velocity and acceleration of a body moving in a plane resolved along and perpendicular to the radius vector.*

* Cf. Tait's Quat., Art. 337.

Having thus, by a succession of operations comprised well within the domain of algebra, and with the aid of notations borrowed from Quaternions, obtained an operator which rotates a vector through a definite angle about a definite axis, I will now pause to compare q^ϑ with the unit versor $\varepsilon^{\frac{2\vartheta}{\pi}}$ of quaternions.*

$\varepsilon^{\frac{2\vartheta}{\pi}}$ and q^ϑ have precisely the same effect on a vector in the plane perpendicular to their axes; but $\varepsilon^{\frac{2\vartheta}{\pi}}$ operating on ε gives $\varepsilon^{\frac{2\vartheta+\pi}{\pi}}$, a versor whose angle exceeds that of $e^{\frac{2\vartheta}{\pi}}$ by $\frac{\pi}{2}$, its plane being the same; while q^ϑ operating on its axis v' simply leaves it unchanged. For this reason the effects of these operators on vectors not perpendicular to their axes are different, that of q^ϑ being the simpler. We may also compare q^ϑ with the quaternion operator $q.\alpha q^{-1}$ which rotates α about the axis and through double the angle of q.†

Returning to equation (41) and observing that if

$$\delta = a\alpha + b\beta + c\gamma,\ \delta' = a'\alpha + b'\beta + c'\gamma, \tag{50}$$

then $$V(\delta,\delta') = (bc' - cb')\alpha + (ca' - ac')\beta + (ab' - ba')\gamma,$$

so that

$$V(\rho + 2cy'^2\rho, v') = (2x - y - z)\alpha + (2y - z - x)\beta + (2z - y - x)\alpha;$$

and writing the equation

$$3\mu\rho = v'S.v'\rho + \cos\phi\, V(\rho + 2cy'^2\rho, v') + \sqrt{3}\sin\phi\, V(\rho, v'), \tag{51}$$

we have, if ϕ is a very small angle, and if we note that $v'.Sv'\rho = (x + y + z)(\alpha + \beta + \gamma)$,

$$3\mu\rho = 3\rho + \sqrt{3}\,d\phi\, V(\rho, v'), \tag{52}$$

to quantities of the second order.

For a rotation of σ through an angle $d\vartheta$ about an axis v'', which meets v', we have

$$3\mu'\sigma = 3\sigma + \sqrt{3}\,d\vartheta\, V(\sigma, v'').$$

In this writing $\mu\rho$ for σ, we shall have the effect of the two rotations in succession upon ρ. This gives

$$\left.\begin{aligned} 9\mu'\mu\rho &= 3(3\rho + \sqrt{3}\,d\phi\, V(\rho, v') + \sqrt{3}\,d\vartheta\, V(3\rho + \sqrt{3}\,d\phi\, V(\rho, v'), v'')\\ &= 9\rho + 3\sqrt{3}\,d\phi\, V(\rho, v') + 3\sqrt{3}\,d\vartheta\, V(\rho, v'')\ddagger\\ &= 9q^{d\vartheta}q^{d\phi}\rho. \end{aligned}\right\} \tag{53}$$

Here the order of the rotations is indifferent, but that is of course not true for finite rotations, μ' being in general a function of $\mu\rho$.

*Tait, p. 88; Ex. 13. † Tait, p. 261. ‡ Compare Tait, p. 260.

4.—*Applications to Trigonometry.*

The determinant

$$\begin{vmatrix} cy\alpha & cy\alpha & cy^2\alpha \\ cy\beta & cy\beta & cy^2\beta \\ cy\gamma & cy\gamma & cy^2\gamma \end{vmatrix}$$

is identically zero. Hence we conclude that for any three vectors whatever

$$cy\alpha\, V(\beta,\gamma) + cy\beta\, V(\gamma,\alpha) + cy\gamma\, V(\alpha,\beta) = 0.$$

If α, β, γ lie in a plane and their lengths are a, b, c and the angles between them A, B and C, then $V(\beta,\gamma) = bcv' \sin A$, $V(\gamma,\alpha) = cav' \sin\beta$, $V(\alpha,\beta) = abv' \sin C$, v' being the direction of the perpendicular on the plane, and equation (50) becomes, after dividing by v',

$$bc \sin A\, cy\alpha + ca \sin B\, cy\beta + ab \sin C\, cy\gamma = 0;$$

operating on this with cy^2, we have, after dividing by abc, and denoting the versors* of α, β, γ by α', β', γ',

$$\alpha' \sin A + \beta' \sin B + \gamma' \sin C = 0, \tag{55}$$

which shows that if the sides are proportional to the sines of the opposite angles, the plane figure will be closed and a triangle. If $\alpha + \beta + \gamma = 0$ correspond to any other closed figure formed by multiples of α', β', γ', it must be a consequence of (51); otherwise we could eliminate either vector and must conclude that the remaining two are parallel.†

Again, from the equation

$$-\alpha = \beta + \gamma, \tag{56}$$

we have, by squaring and taking scalars,

$$S.\alpha^2 = S.\beta^2 + S.\gamma^2 + 2S.\beta\gamma, \tag{57}$$

or

$$a^2 = b^2 + c^2 - 2bc \cos A. \tag{58}$$

The use of the negative sign in the last term is necessary from the manner of measuring the angle A.

With a spherical triangle we may proceed as follows: We have, if α, β, γ are unit vectors to the vertices,

$$S.V(\alpha,\beta)\, V(\beta,\gamma) = -\cos B \sin a \sin c.$$

*Tait, p. 83. † Cf. Gibbs, Vector Analysis, p. 49.

Now

$$V(\alpha,\beta)\,V(\beta,\gamma)=(cy\alpha cy^2\beta-cy\beta cy^2\alpha)(cy\beta cy^2\gamma-cy\gamma cy^2\beta)$$
$$=-cy\beta^2cy^2(\alpha\gamma)+cy(\beta\gamma)\,cy^2(\alpha\beta)+cy(\alpha\beta)\,cy^2(\beta\gamma)-cy(\alpha\gamma)\,cy^2\beta^2;$$

and recalling that

$$S.cy(\alpha\beta)\,cy^2(\beta\gamma)=S.\beta\gamma cy^2(\alpha\beta)\ \text{ and }\ S.cy(\beta\gamma)\,cy^2(\alpha\beta)=S.\beta\gamma cy(\alpha\beta),$$

by a formula of Art. 2, we may write this

$$\begin{aligned}S.V(\alpha,\beta)\,V(\beta,\gamma)&=S.\beta\gamma\left(cy(\alpha\beta)+cy^2(\alpha\beta)\right)-S.\beta^2\left(cy(\alpha\gamma)+cy^2(\alpha\gamma)\right)\\&=S.\beta\gamma\,(vS.\alpha\beta-\alpha\beta)-S.\beta^2\,(vS.\alpha\gamma-\alpha\gamma)\\&=S.\beta\gamma S.\alpha\beta-S.\beta^2S.\alpha\gamma\\&=\cos a\cos c-\cos b,\end{aligned}$$

by previous formulae. We conclude that

$$\cos b=\cos a\cos c+\sin a\sin c\cos B.^*$$

This extremely simple process may be compared with the analysis given in Tait, p. 56, Art. 90, which must be read to get the result on p. 71.

Without following so closely the familiar processes of quaternions, we may proceed as follows:

$$\begin{aligned}V(\alpha,\beta)\,V(\beta,\gamma)&=\begin{vmatrix}cy\alpha & cy\beta\\ cy^2\alpha & cy^2\beta\end{vmatrix}\cdot\begin{vmatrix}cy\beta & cy\gamma\\ cy^2\beta & cy^2\gamma\end{vmatrix}\\&=\begin{vmatrix}cy(\alpha\beta)+cy^2(\alpha\beta), & cy(\alpha\gamma)+cy^2(\alpha\gamma)\\ cy\beta^2+cy^2\beta^2\quad, & cy(\beta\gamma)+cy^2(\beta\gamma)\end{vmatrix}\\&=\begin{vmatrix}vS.\alpha\beta-\alpha\beta, & vS.\alpha\gamma-\alpha\gamma\\ vS.\beta^2-\beta^2\ , & vS.\beta\gamma-\beta\gamma\end{vmatrix}\\&=vS.\alpha\beta S.\beta\gamma-\beta\gamma S.\alpha\beta-\alpha\beta S.\beta\gamma-vS.\beta^2S.\alpha\gamma\\&\quad+\alpha\gamma S.\beta^2+\beta^2S.\alpha\gamma.\end{aligned}$$

Hence, taking scalars, we have

$$S.\alpha\beta S.\beta\gamma-S.\beta^2S.\alpha\gamma=S.V(\alpha,\beta)\,V(\beta,\gamma)$$

as before, by remembering that $S.v=3$.

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* Tait, p. 71.

Sur une forme nouvelle de l'équation modulaire du huitième dégré.

PAR FRANCESCO BRIOSCHI.

1. Soit: $$f(x)=x^3+3a_1x^2+3a_2x+a_3=0$$
l'équation dont les racines sont:

$$x_0=\wp\left(\frac{2\omega}{7}\right),\ x_1=\wp\left(\frac{4\omega}{7}\right),\ x_2=\wp\left(\frac{6\omega}{7}\right)=\wp\left(\frac{8\omega}{7}\right).$$

Comme il est connu, entre les trois coefficients a_1, a_2, a_3 on a trois relations desquelles par l'elimination de deux des ces coefficients on obtient pour le troisième une équation modulaire du huitième dégré. On sait encore que en indiquant avec D le symbole d'operation:

$$D=12g_3\frac{d}{dg_2}+\frac{2}{3}g_2^2\frac{d}{dg_3}$$

on a:*

$$\left.\begin{aligned} 7D(a_1)&=20a_2-18a_1^2+\frac{13}{6}g_2,\\ 7D(a_2)&=14a_3-18a_1a_2+\frac{19}{3}g_2a_1-2g_3,\\ 7D(a_3)&=\quad\ -18a_1a_3+\frac{25}{2}g_2a_2-6g_3a_1.\end{aligned}\right\}\qquad(1)$$

La somme des arguments des trois racines x_0, x_1, x_2 étant une période, on peut poser:

$$a_2=-2a_1\xi-\frac{1}{12}g_2,\quad a_3=3a_1\xi^2-\frac{1}{4}g_3\qquad(2)$$

qui donnent pour $f(x)$ la valeur:

$$f(x)=x^3-\frac{1}{4}g_2x-\frac{1}{4}g_3+3a_1(x-\xi)^2.$$

* Voir Comptes Rendus de l'Académie des Sciences, Janvier 1891.

Operant avec D sur les deux relations (2) on aura pour les précédentes (1):

$$\left.\begin{aligned} 14a_1D(\xi) &= 38a_1\xi^3 - \frac{47}{6}g_2a_1 - g_2\xi - \frac{3}{2}g_3, \\ 14a_1\xi D(\xi) &= 40a_1\xi^3 - \frac{25}{3}g_2a_1\xi - \frac{1}{2}g_2\xi^2 - \frac{1}{2}g_3a_1 + \frac{1}{24}g_2^2 \end{aligned}\right\} \quad (3)$$

desquelles en eliminant $D(\xi)$ on obtient pour a_1 la valeur:

$$a_1\left(\xi^3 - \frac{1}{4}g_2\xi - \frac{1}{4}g_3\right) = -\frac{1}{4}\left[g_2\xi^2 + 3g_3\xi + \frac{1}{12}g_2^2\right].$$

Or en posant:

$$c_1 = \xi,\ c_2 = \xi^2 - \frac{1}{12}g_2,\ c_3 = \xi^3 - \frac{1}{4}g_2\xi - \frac{1}{4}g_3,$$

$$c_4 = \xi^4 - \frac{1}{2}g_2\xi^2 - g_3\xi - \frac{1}{48}g_2^2 = 4c_1c_3 - 3c_2^2$$

on déduit:

$$\left.\begin{aligned} &\frac{1}{12}g_2 = c_1^2 - c_2,\quad \frac{1}{4}g_3 = -2c_1^3 + 3c_1c_2 - c_3, \\ &\frac{1}{4^2.3^3}\delta = 3c_1^2c_2^2 + 6c_1c_2c_3 - 4c_1^3c_3 - 4c_2^3 - c_3^2 \end{aligned}\right\} \quad (4)$$

étant $\delta = g_2^3 - 27g_3^2$; et en conséquence:

$$\left.\begin{aligned} a_1 &= 3\frac{c_1c_3 - c_2^2}{c_3} = \frac{3}{4}\frac{c_4 - c_2^2}{c_3}, \\ a_2 &= \frac{1}{c_3}\left[6c_1c_2^2 - 7c_1^2c_3 + c_2c_3\right], \\ a_3 &= \frac{1}{c_3}\left[11c_1^3c_3 - 9c_1^2c_2^2 - 3c_1c_2c_3 + c_2^3\right]. \end{aligned}\right\} \quad (5)$$

La prémière des équations (3) donne de la même manière:

$$D(c_1) = \frac{2}{7(c_4 - c_2^2)}\left[14c_1^2c_2^2 - 14c_1^2c_4 + 23c_2c_4 - 25c_2^3 + 2c_3^2\right]$$

et l'on déduit:

$$D(c_2) = \frac{2}{7(c_4 - c_2^2)}\left[4c_1c_2c_4 - 8c_1c_2^3 + 15c_3c_4 - 11c_2^2c_3\right],$$

$$D(c_3) = \frac{3}{7(c_4 - c_2^2)}\left[7c_4^2 + 4c_2^2c_4 - 15c_2^4 + 4c_2c_3^2\right]$$

et

$$28c_3^2(c_4 - c_2^2)D\left[\frac{c_1c_3 - c_2^2}{c_3}\right] = -7c_4^3 + 33c_4^2c_2^2 + 11c_4c_2^4 - 21c_2^6 - 56c_2c_3^2c_4 + 24c_2^3c_3^2 + 16c_3^4.$$

D'autre part de la prémière des relations (1) on obtient:

$$\frac{28}{3}\, c_3^2 D(a_1) = -23c_4^2 + 10c_2^2 c_4 + 21c_2^4 - 8c_2c_3^2$$

le premier membre de laquelle, multiplié par $c_4 - c_2^2$, étant pour la valeur (5) de a_1 égal au premier membre de la précédente, en egalant les secondes membres on arrive *à cette forme nouvelle de l'équation modulaire du huitième dégré:*

$$c_4^3 + c_3^4 + c_2^3 c_3^2 - 3c_2 c_3^2 c_4 = 0$$

ou:

$$c_4 + c_2 c_3^{\frac{2}{3}} + c_3^{\frac{4}{3}} = 0. \tag{6}$$

2. L'équation (6) est satisfaite en posant:

$$c_2 = \frac{1}{3}\, c_3^{\frac{2}{3}}(z^4 + 5), \quad c_4 = -\frac{1}{3}\, c_3^{\frac{4}{3}}(z^4 + 8) \tag{7}$$

desquelles, étant $c_4 = 4c_1c_3 - 3c_2^2$, on deduit:

$$c_1 = \frac{1}{12}\, c_3^{\frac{1}{3}}(z^8 + 9z^4 + 17). \tag{8}$$

Substituant ces valeurs dans les trois équations (4), on obtient:

$$\left.\begin{array}{l} g_2 = \frac{1}{12}\, c_3^{\frac{2}{3}} PQ, \quad g_3 = -\frac{1}{216}\, c_3 PR, \\ \delta = c_3^2 z^4 P^2 \end{array}\right\} \tag{9}$$

étant

$$P = z^8 + 13z^4 + 49, \quad Q = z^8 + 5z^4 + 1,$$
$$R = z^{16} + 14z^{12} + 63z^8 + 70z^4 - 7$$

et en conséquence:

$$\frac{g_2^3}{8} = T = \frac{1}{12^{-3}}\, \frac{PQ^3}{z^4}, \quad T - 1 = \frac{1}{12^{-3}}\, \frac{R^2}{z^4} \tag{10}$$

formules connues.

Par la même substitution les équations (5) donnent:

$$\left.\begin{array}{l} a_1 = -\frac{1}{12}\, c_3^{\frac{1}{3}} P, \quad a_2 = \frac{1}{12^{-2}}\, c_3^{\frac{2}{3}} P(P - 16), \\ a_3 = -\frac{1}{12^{-3}}\, c_3 P\,[P^2 - 32P - 16Q + 64] \end{array}\right\} \tag{11}$$

et en indiquant avec d le discriminant de $f(x)$, ou:

$$\frac{1}{27}\,d = 3a_1^2a_2^2 + 6a_1a_2a_3 - 4a_1^3a_3 - 4a_2^3 - a_3^2$$

on trouve:
$$d = c_3^2 P^2$$

ou par la valeur δ (9):

$$d = \frac{\delta}{z^4}$$

et
$$z^3 = \frac{\delta^{\frac{1}{2}}}{(x_0 - x_1)(x_1 - x_2)(x_2 - x_0)}\,.$$

J'ai déjà rappellé que pour chacun des coefficients a_1, a_2, a_3 on a une équation du huitième dégré. Si l'on pose:

$$h_1 = -a_1,\quad h_2 = h_1^2 - \frac{1}{12}\,g_2,\quad h_3 = h_1^3 - \frac{1}{4}\,g_2h_1 - \frac{1}{4}\,g_3,\quad h_4 = 4h_1h_3 - 3h_2^2$$

de la valeur (11) de a_1 on deduit:

$$h_1 = \frac{1}{12}\,c_3^{\frac{1}{3}}P,\quad h_2 = \frac{1}{12^2}\,c_3^{\frac{2}{3}}P\,(P - Q),\quad h_3 = \frac{c_3P}{12^3}\,(P^2 - 3PQ + 2R)$$
$$h_4 = \frac{c_3^{\frac{4}{3}}P^2}{12^4}\,[4P^2 - 12PQ + 8R - 3\,(P - Q)^2],$$

mais:
$$4P^2 - 12PQ + 8R - 3\,(P - Q)^2 = 32\,(P - Q + 16) = 4^4\,(z^4 + 8)$$

et en conséquence (7)
$$27h_4 = -P^2c_4. \tag{12}$$

En second lieu étant $P - Q - 8 = 8\,(z^4 + 5)$ on aura:

$$6\,(h_2 - m) = Pc_2 \tag{13}$$

étant $m = \frac{1}{18}\,c_3^{\frac{2}{3}}P$.

En multipliant par P^2 l'équation modulaire (6) on obtient une première relation:

$$8m^2 + 4h_2m - h_4 = 0.$$

La valeur (5) de a_1 donne:

$$4h_1c_3 \quad = 3\,(c_2^2 - c_4)$$

et
$$4h_1c_3P^2 = 108\,(h_2 - m)^2 + 81h_4,$$

mais
$$c_3P^2 = 18\,.\,12\,.\,mh_1 \tag{14}$$

et on arrive ainsi à une seconde relation,

$$4m^2 - 8(h_2 + 4h_1^2)\,m + 4h_2^2 + 3h_4 = 0$$

et par l'elimination de m à l'équation modulaire en h_1:

$$2h_4(5h_2 + 16h_1^2)^2 - 2h_2(5h_2 + 16h_1^2)(8h_2^2 + 7h_4) - (8h_2^2 + 7h_4)^2 = 0 \tag{15}$$

ou à l'équation connue:

$$h^8 - \frac{7}{3}\,g_2h^6 - 14g_3h^5 - \frac{35}{24}\,g_2^2h^4 - \frac{7}{3}\,g_2g_3h^3 - \frac{7}{3^6}\left(\frac{23}{16}\,g_2^3 + 7.3^3.g_3^2\right)h^2$$
$$- \frac{1}{24}\,g_2^2g_3h - \frac{7}{12^{-4}}\,g_2^4 = 0$$

ayant posé h au lieu de h_1.

On ait ainsi conduits aux deuxé quations modulaires connues comme conséquences de l'équation (6).

3. Les inconnues dans ces trois équations sont $\xi = c_1,\ h,\ z.(6,\ 10,\ 15)$. Des formules supérieures on deduit facilement les relations existantes entre ces inconnues.

En effet la prémière des équations (5) donne:

$$c_1 + a_1 = \frac{c_4}{c_3}$$

et à cause de (12), (14):

$$c_1 = h_2 - \frac{h_4}{8mh_1},$$

mais de la prémière des équations on a:

$$\frac{h_4}{8m} = m + \frac{1}{2}\,h_2$$

donc

$$c_1 = \frac{1}{2h_1}\,(2h_1^2 - h_2 - 2m).$$

Or en posant:

$$\lambda = 8h_2^2 + 7h_4, \quad \mu = 5h_2 + 16h_1^2$$

on déduit des deux équations en m:

$$4m = \frac{\lambda}{\mu}$$

et la (15) peut s'écrire

$$\lambda^2 + 2h_2\lambda\mu - 2h_4\mu^2 = 0.$$

On aura donc:
$$c_1 = -\frac{1}{4h_1\mu}\left[2(2h_1^2 - h_2)\mu - \lambda\right]$$
ou
$$c_1 = \frac{1}{4h_1\mu}\left[64h_1^4 - 12h_1^2h_2 - 18h_2^2 - 7h_4\right].$$

La relation cherchée entre z, h s'obtient en observant que des équations (7) on deduit:
$$8c_2c_3^2 + 5c_4 = c_3^2z^4$$
par conséquent si on multiplie par P^2 on aura
$$32m(h_2 - m) - 5h_4 = 12m^2z^4,$$
mais la prémière des équations en m donne:
$$h_4 = 8m^2 + 4h_2m$$
ainsi
$$z^4 = \frac{h_2 - 6m}{m}$$
ou
$$z^4 = \frac{4h_2\mu - 6\lambda}{\lambda} = \frac{2}{\lambda}\left[32h_1^2h_2 - 14h_2^2 - 21h_4\right].$$

MILAN, *Mars* 1891.